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Estimates for Boundary Blowup Solutions of p-Laplacian Type Quasilinear Elliptic Equations

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Abstract

In this paper, we investigate the effect of the mean curvature of the boundary $\partial\Omega$ on the behavior of the blow-up solutions to the p-Laplacian type quasilinear elliptic equation

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = u^m|\nabla u|, \quad p > 1,$$

where the $\Omega \in R^N$ be a bounded smooth domain. Under appropriate conditions on p and m, we find the estimates of the solution u interms of the distance from x to the boundary $\partial\Omega$. To the equation

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = u^m|\nabla u|^q, \quad p > 1, \quad 0 < q < 1,$$

the results of the semilinear problem are extended to the quasilinear ones.

Keywords: p-Laplacian elliptic equation; boundary blow-up solution; estimates.

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1 Introduction

In this paper, we study the boundary blow-up problems

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u^m|\nabla u| \ \text{ in } \Omega, \ \ u \to \infty \ \text{ as } \ x \to \partial\Omega, \tag{1.1}$$

and

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u^m|\nabla u|^q \ \text{ in } \Omega, \ \ u \to \infty \ \text{ as } \ x \to \partial\Omega, \tag{1.2}$$

where Ω is a bounded smooth domain in R^N, $N \geq 2$, $p > 1$, $m+1 > p-1$, and $0 < q < 1$.

First we consider to prove the existence of a positive large solution. We first consider, for $0 < \varepsilon < 1$, the problem

$$\Delta_p u = u^m(\varepsilon + |\nabla u|^2)^{\frac{1}{2}} \ \text{ in } \Omega, \ \ u \to \infty \ \text{ as } \ x \to \infty,$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The existence of a positive solution $u = u_\varepsilon$ for this new problem is proved in [1], [2], [3], [4]. Then, by theorem 4.2 of [4] a sequence u_{ε_i}, with $\varepsilon_i \to 0$, tends to a solution u of problem (1.1).

We are interesting in the behavior of the solution u near the boundary $\partial\Omega$. Problems of this kind are discussed in many papers, see, for instance, [5], [6], [7], [8], [9] and the survey paper [10]. Some papers found some estimates, such as [11], [12]. For the problem

$$\Delta u = u^m \ \text{ in } \Omega, \ \ u \to \infty \ \text{ as } \ x \to \infty. \tag{1.3}$$

C.Bandle in [8] has found the estimate

$$u(x) = \left(\frac{p-1}{\sqrt{2(p+1)}}\delta(x)\right)^{\frac{2}{1-p}} = \left[1 + \frac{(N-1)H(\overline{x})}{p+3} + o(\delta(x))\right], \tag{1.4}$$

where $\delta(x)$ denotes the distance from x to the boundary $\partial\Omega$, and $H(\overline{x})$ denotes the mean curvature of $\partial\Omega$ at the point $\overline{x}$ nearest to x.

In [11], the authors investigate the problem

$$\Delta u = u^p|\nabla u|^q \ \text{ in } \Omega, \ \ u \to \infty \ \text{ as } \ x \to \infty. \tag{1.5}$$

where Ω is a bounded smooth domain in R^N, $N \geq 2$, $p > 0$, $0 \leq q \leq (p+3)/(p+2)$ and $p+q > 1$. They find an estimate similar to (1.4).

More precisely, let $A(\rho, R) \subset R^N$, $N \geq 2$, be the annulus with radius ρ and R centered at the origin, $u(x)$ be a radial solution to problem (1.5) in $\Omega = A(\rho, R)$, and let $v(r) = u(x)$ for $r = |x|$. If $p > 0$, $0 \leq q < (p+3)/(p+2)$ and $p+q > 1$ they have

$$v(r) < \phi(R-r)[1 + C(R-r)], \ \ r \in (r_1, R), \tag{1.6}$$

$$v(r) > \phi(r-\rho)[1 - C(r-\rho)], \ \ r \in (\rho, r_2). \tag{1.7}$$

where ϕ be the function defined by

$$\phi(t) = \left(\frac{2-q}{p+q-1}\right)^{\frac{2-q}{p+q-1}} \left(\frac{p+1}{2-q}\right)^{\frac{1}{p+q-1}} t^{\frac{q-2}{p+q-1}}. \tag{1.8}$$

and r_1 is a constant between r_0 and R, r_2 is a constant between ρ and r_0.

If $p > 0$, $q = (p+3)/(p+2)$ they have

$$v(r) < \phi(R-r)[1 + C(R-r)\ln\frac{1}{R-r}], \ \ r \in (r_1, R), \tag{1.9}$$

$$v(r) > \phi(r-\rho)[1 - C(r-\rho)\ln\frac{1}{r-\rho}],\ \ r \in (\rho, r_2). \tag{1.10}$$

Let Ω be a bounded domain with a smooth boundary $\partial\Omega$, let $p > 0,\ 0 \le q < (p+3)/(p+2)$ and $p + q > 1$, they have

$$v(x) \le u(x) \le w(x),$$

where ϕ be the function defined in (1.8), and

$$w(x) = \phi(\delta)\left(1 + \frac{(2-q)(N-1)H(x)}{2(p+3-q(p+2))}\delta + \alpha\delta^{\sigma}\right), \tag{1.11}$$

$$v(x) = \phi(\delta)\left(1 + \frac{(2-q)(N-1)H(x)}{2(p+3-q(p+2))}\delta - \alpha\delta^{\sigma}\right). \tag{1.12}$$

Motivated by the results of the above cited papers, we further study the estimates for boundary blow-up solutions of problem (1.1)-(1.2), the partial results of the semilinear problem are extended to the quasilinear ones. We can find the related part results for $p = 2$ in [1].

2 Estimates for Radial Solution

In this section, firstly, we study the problem (1.1), we present some lemmas that will be used in the section.

Lemma 2.1. Let $p > 0, m + 1 > p - 1$. Consider the equation in (1.1) in dimension $N = 1$ and $\Omega = (0, \infty)$. If $u = \phi(t) > 0$ and $\phi'(t) < 0$ we have

$$\phi''(-\phi')^{p-2} = \phi^m(-\phi'). \tag{2.1}$$

where $\phi(t)$ be defined by

$$\phi(t) = (m+1)^{\frac{1}{(m+1)-(p-1)}}\left[\frac{p-1}{(m+1)-(p-1)}\right]^{\frac{p-1}{(m+1)-()p-1}} t^{\frac{1-p}{(m+1)-(p-1)}}. \tag{2.2}$$

A solution of (2.1) such that $\phi(t) \to \infty\ \ as\ \ t \to 0$ is precisely the function defined in (2.2).

In what follows we denote by $C > 1$ a constant which may change from term to term.

Lemma 2.2[14]. Let $g(r)$ be a C^1-function defined for $R_1 < r < R$. If $g(r) \to \infty\ \ as\ \ r \to R_1^+$ and $g'(r) \le 0$, then: $\lim\limits_{r\to R_1^+} \frac{\int_r^R g(s)ds}{g(r)} = 0$.

Theorem 2.1. Let $A(\rho, R) \subset R^N,\ \ N \ge 2$, be the annulus with radii ρ and R centered at the origin. Let ϕ be the function defined in (2.2), let $u(x)$ be a radial solution to problem (1.1) in $A(\rho, R) \subset R^N$, and let $v(r) = u(x)$ for $r = |x|$. If $p > 0,\ \ m + 1 > p - 1$ we have

$$v(r) < \phi(R-r)[1 + C(R-r)],\ \ r \in (r_1, R), \tag{2.3}$$

$$v(r) > \phi(r-\rho)[1 - C(r-\rho)],\ \ r \in (\rho, r_2). \tag{2.4}$$

Proof. If $\Omega = A(\rho, R)$, problem 1.1 reads as

$$\left(|v'|^{p-2}v'\right)' + \frac{N-1}{r}|v'|^{p-2}v' = v^m|v'|,\ \ v(\rho) = v(R) = \infty. \tag{2.5}$$

There is a point $r_0 \in (\rho, R)$ such that $v'(r_0) = 0,\ v'(r) < 0$ for $r \in (\rho, r_0)$ and $v'(r) > 0$ for $r \in (r_0, R)$. For $r \in (r_0, R)$ we have

$$\left((v')^{p-1}\right)' + \frac{N-1}{r}(v')^{p-1} = v^m v',\ \ v'(r_0) = 0,\ \ v(R) = \infty. \tag{2.6}$$

Integration over (r_0, r) yields

$$(v')^{p-1}\big|_{r_0}^{r} + \int_{r_0}^{r} \frac{N-1}{s}(v')^{p-1}ds = \int_{r_0}^{r} v^m v' ds,$$

$$(v')^{p-1}\big|_{r_0}^{r} + (N-1)\int_{r_0}^{r} \frac{(v')^{p-1}}{s}ds = \frac{v^{m+1}-v_0^{m+1}}{m+1}, \ \ v_0 = v(r_0),$$

$$(v')^{p-1} + (N-1)\int_{r_0}^{r} \frac{(v')^{p-1}}{s}ds = \frac{v^{m+1}-v_0^{m+1}}{m+1}. \tag{2.7}$$

From (2.7) we find

$$(v')^{p-1} < \frac{v^{m+1}}{m+1},$$

$$v' < \frac{v^{\frac{m+1}{p-1}}}{m+1}.$$

On the other hand, by lemma 2.2 we have

$$\lim_{r\to R} \frac{(v')^{p-1}}{\int_{r_0}^{r} \frac{(v')^{p-1}}{s}} = \infty.$$

and combining this with (2.7) implies for $r \in (r_1, R)$

$$2(v')^{p-1} > \frac{v^{m+1}}{m+1}.$$

Hence by Eq.(2.7) we find

$$\frac{1}{C}v^{\frac{m+1}{p-1}} < v' < Cv^{\frac{m+1}{p-1}}, \ \ r \in (r_1, R), \tag{2.8}$$

From (2.8) we find

$$\frac{1}{C}v^{\frac{m+1}{1-p}} < \frac{1}{v'} < Cv^{\frac{m+1}{1-p}},$$

$$\frac{1}{C}\int_{r}^{R} v^{\frac{m+1}{1-p}} v' ds < R - r < C\int_{r}^{R} v^{\frac{m+1}{1-p}} v' ds,$$

$$\frac{1}{C}\left(0 - v^{\frac{m+1-(p-1)}{1-p}}\right) < R - r < C\left(0 - v^{\frac{m+1-(p-1)}{1-p}}\right),$$

$$\frac{1}{C}v^{\frac{m+1-(p-1)}{1-p}} < R - r < Cv^{\frac{m+1-(p-1)}{1-p}},$$

Finally we get

$$\frac{1}{C}(R-r)^{\frac{1-p}{(m+1)-(p-1)}} < v < C(R-r)^{\frac{1-p}{(m+1)-(p-1)}}, \tag{2.9}$$

and

$$\frac{1}{C}(R-r)^{\frac{m+1}{(p-1)-(m+1)}} < v' < C(R-r)^{\frac{m+1}{(p-1)-(m+1)}}. \tag{2.10}$$

By using (2.10), we find

$$\begin{aligned} \int_{r_0}^{r} \frac{(v')^{p-1}}{s}ds \quad & < \int_{r_0}^{r} \frac{C^{p-1}(R-s)^{\frac{(m+1)(p-1)}{(p-1)-(m+1)}}}{s}ds \\ & < \frac{C^{p-1}}{r_0}\int_{r_0}^{r}(R-s)^{\frac{(m+1)(p-1)}{(p-1)-(m+1)}}ds \\ & < C(R-r)^{\frac{(m+2)(p-1)-(m+1)}{(p-1)-(m+1)}} \end{aligned} \tag{2.11}$$

Inserting estimate (2.11) into (2.7) we get

$$(v')^{p-1} > \frac{v^{m+1}-v_0^{m+1}}{m+1} - C(R-r)^{\frac{(m+2)(p-1)-(m+1)}{(p-1)-(m+1)}},$$

$$(m+1)\frac{(v')^{p-1}}{v^{m+1}} > 1 - \frac{C(m+1)(R-r)^{\frac{(m+2)(p-1)-(m+1)}{(p-1)-(m+1)}} + v_0^{p+1}}{v^{m+1}}.$$

From (2.9) we get

$$(m+1)\frac{(v')^{p-1}}{v^{m+1}} > 1 - C(R-r),$$

$$(m+1)^{\frac{1}{p}}\frac{v'}{v^{\frac{m+1}{p-1}}} > 1 - C(R-r).$$

Integration over (r, R) yields

$$(m+1)^{\frac{1}{p-1}}\frac{p-1}{(p-1)-(m+1)}v^{\frac{(p-1)-(m+1)}{p-1}} > R - r - C(R-r)^2,$$

$$v^{\frac{(p-1)-(m+1)}{p-1}} < (m+1)^{\frac{1}{p-1}}\frac{(p-1)-(m+1)}{p-1}(R-r)[1-C(R-r)],$$

$$v(r) < (m+1)^{\frac{1}{(m+1)-(p-1)}}\left[\frac{p-1}{(m+1)-(p-1)}\right]^{\frac{p-1}{(m+1)-(p-1)}}(R-r)^{\frac{1-p}{(m+1)-(p-1)}}[1-C(R-r)]^{\frac{1-p}{(m+1)-(p-1)}}.$$

Since

$$(1-C(R-r))^{\frac{1-p}{(m+1)-(p-1)}} < 1 + C(R-r),$$

with a new constant C, we get

$$v(r) < \phi(r)[1 + C(R-r)],$$

where ϕ be the function defined by (2.2).

Let us prove inequality (2.4). For $r \in (\rho, r_0)$ we have $v'(r) < 0$, and

$$\left((-v')^{p-2}v'\right)' - \frac{N-1}{r}(-v')^{p-1} = -v^m v', \quad v(\rho) = \infty, \quad v'(r_0) = 0. \tag{2.12}$$

Integration over (r, r_0) yields

$$(-v')^{p-2}v'\Big|_{r_0}^{r} - (N-1)\int_{r_0}^{r}\frac{(-v')^{p-1}}{s}ds = -\frac{v^{m+1}}{m+1}\Big|_{r}^{r_0}, \quad v_0 = v(r_0),$$

$$0 - (-v')^{p-2}v' - (N-1)\int_{r}^{r_0}\frac{(-v')^{p-1}}{s}ds = \frac{v^{m+1} - v_0^{m+1}}{m+1},$$

$$(-v')^{p-1} - (N-1)\int_{r}^{r_0}\frac{(-v')^{p-1}}{s}ds = \frac{v^{m+1} - v_0^{m+1}}{m+1}. \tag{2.13}$$

Arguing as in the precious case, now we find

$$\frac{1}{C}v^{\frac{m+1}{p-1}} < -v' < Cv^{\frac{m+1}{p-1}}, \quad r \in (\rho, r_2),$$

so

$$\frac{1}{C}(r-\rho)^{\frac{1-p}{(m+1)-(p-1)}} < v < C(r-\rho)^{\frac{1-p}{(m+1)-(p-1)}}, \tag{2.14}$$

and

$$\frac{1}{C}(r-\rho)^{\frac{m+1}{(p-1)-(m+1)}} < v' < C(r-\rho)^{\frac{m+1}{(p-1)-(m+1)}}. \tag{2.15}$$

By using (2.15), we find

$$\begin{aligned}\textstyle\int_r^{r_0}\frac{(-v')^{p-1}}{s} &< \textstyle\int_r^{r_0}\frac{C^{p-1}(s-\rho)^{\frac{(m+1)(p-1)}{(p-1)-(m+1)}}}{s}ds \\ &< C(r-\rho)^{\frac{(m+2)(p-1)-(m+1)}{(p-1)-(m+1)}}.\end{aligned} \tag{2.16}$$

Inserting estimate (2.16) *into* (2.13) we get

$$(-v')^{p-1} < \frac{v^{m+1} - v_0^{m+1}}{m+1} + C(r-\rho)^{\frac{(m+2)(p-1)-(m+1)}{(p-1)-(m+1)}},$$

$$(m+1)\frac{(-v')^{p-1}}{v^{m+1}} < 1 + \frac{C(m+1)(r-\rho)^{\frac{(m+2)(p-1)-(m+1)}{(p-1)-(m+1)}} - v_0^{p+1}}{v^{m+1}}.$$

From (2.14) we get

$$(m+1)\frac{(-v')^{p-1}}{v^{m+1}} < 1 + C(r-\rho),$$

$$(m+1)^{\frac{1}{p-1}}\frac{-v'}{v^{\frac{m+1}{p-1}}} < 1 + C(r-\rho).$$

Integration over (ρ, r) yields

$$(m+1)^{\frac{1}{p-1}}\frac{p-1}{(m+1)-(p-1)}v^{\frac{(p-1)-(m+1)}{p-1}} < (r-\rho) + C(r-\rho)^2,$$

$$(m+1)^{\frac{1}{p-1}}\frac{p-1}{(m+1)-(p-1)}\left[(r-\rho)\left(1+C(r-\rho)\right)\right]^{-1} < v\frac{(m+1)-(p-1)}{p-1},$$

$$v(r) \quad > \quad (m+1)^{\frac{1}{(m+1)-(p-1)}}\left[\frac{p-1}{(m+1)-(p-1)}\right]^{\frac{p-1}{(m+1)-(p-1)}}(r-\rho)^{\frac{1-p}{(m+1)-(p-1)}}$$

$$[1+C(r-\rho)]^{\frac{1-p}{(m+1)-(p-1)}}.$$

Since

$$(1+C(r-\rho))^{\frac{1-p}{(m+1)-(p-1)}} > 1 - C(r-\rho),$$

we get

$$v(r) > \phi(r)[1 - C(r-\rho)].$$

where $\phi(r)$ be the function defined by (2.2).

The theorem is proved.

Now let us investigate the problem (1.2). If $p > 1, 0 < q < 1$, and $m+q > p-1$, we can get sectional similar arguments as follow.

Let $p > 1, 0 < q < 1$, and $m + q > p - 1$. Consider the equation in (1.2) in dimension $N = 1$ and $\Omega = (0, \infty)$. If $u = \phi_1(t) > 0$ and $\phi_1'(t) < 0$ we have

$$\phi_1''(-\phi_1')^{p-2} = \phi_1^m(-\phi_1')^q. \tag{2.17}$$

Where ϕ_1 be defined by

$$\phi_1(t) = (m+1)^{\frac{1}{(m+q)-(p-1)}}\left[\frac{p-q}{(m+q)-(p-1)}\right]^{\frac{p-q}{(m+q)-()p-1}} t^{\frac{q-p}{(m+q)-(p-1)}}. \tag{2.18}$$

Theorem 2.2. Let $A(\rho, R) \subset R^N$, $N \geq 2$, be the annulus with radii ρ and R centered at the origin. Let ϕ_1 be the function defined in (2.17), let $u(x)$ be a radial solution to problem (1.2) in $A(\rho, R) \subset R^N$, and let $v(r) = u(x)$ for $r = |x|$. If $p > 1$, $0 < q < 1$, *and* $m + q > p - 1$ we have

$$v(r) < \phi_1(R-r)[1 + C(R-r)], \quad r \in (r_1, R). \tag{2.19}$$

Proof. If $\Omega = A(\rho, R)$,problem $(1.1a)$ reads as

$$\left(r^{N-1}\phi_p(v')\right)' = r^{N-1}v^m|v'|^q, \quad v(\rho) = v(R) = \infty, \tag{2.20}$$

where $\phi_p(v') = |v'|^{p-2}v'$. There is a point $r_0 \in (\rho, R)$ such that $v'(r_0) = 0$, $v'(r) > 0$ for $r \in (r_0, R)$. For $r \in (r_0, R)$ we have

$$\left(r^{N-1}|v'|^{p-2}v'\right)' = r^{N-1}v^m(v')^q. \tag{2.21}$$

Integration over (r_0, r) yields

$$s^{N-1}\phi_p(v')\Big|_{r_0}^{r} = \int_{r_0}^{r} s^{N-1}v^m(v')^q ds, \quad r \in (r_0, R),$$

$$r^{N-1}\phi_p(v') = \int_{r_0}^{r} s^{N-1}v^m(v')^q ds,$$

$$\phi_p(v') = \frac{1}{r^{N-1}}\int_{r_0}^{r} s^{N-1}v^m(v')^q ds,$$

$$v' = \phi_p^{-1}\left(\frac{1}{r^{N-1}}\int_{r_0}^{r} s^{N-1}v^m(v')^q ds\right),$$

where

$$\phi_p^{-1}(s) = \begin{cases} s^{\frac{1}{p-1}}, & s \geq 0 \\ -(-s)^{\frac{1}{p-1}}, & s < 0. \end{cases}$$

We get

$$v' = \left(\frac{1}{r^{N-1}}\int_{r_0}^{r} s^{N-1}v^m(v')^q ds\right)^{\frac{1}{p-1}}.$$

Since $r \in (r_0, R)$, we find

$$\int_{r_0}^{r} s^{N-1}v^m(v')^q ds \leq R^{N-1}\int_{r_0}^{r} v^m(v')^q ds.$$

From *Hölder* inequality we get

$$\begin{aligned} \textstyle\int_{r_0}^{r} v^m(v')^q ds \quad &< \left(\textstyle\int_{r_0}^{r}((v^m(v')q))^{\frac{1}{q}}ds\right)^q |r - r_0|^{1-q} \\ &= \left(\textstyle\int_{r_0}^{r} v^{\frac{m}{q}}v'ds\right)^q |R - r_0|^{1-q}. \end{aligned} \tag{2.22}$$

From (2.22) we get

$$v' < \left[\frac{R^{N-1}}{r^{N-1}}\left(\int_{r_0}^{r} v^{\frac{m}{q}}v'ds\right)^q (R - r_0)^{1-q}\right]^{\frac{1}{p-1}},$$

$$v' < \left[\frac{R^{N-1}(R - r_0)^{1-q}}{r^{N-1}}\right]^{\frac{1}{p-1}} \left[\left(\int_{r_0}^{r} v^{\frac{m}{q}}v'ds\right)^q\right]^{\frac{1}{p-1}},$$

$$\begin{aligned} v' \quad &< \left[\tfrac{R^{N-1}(R-r_0)^{1-q}}{r^{N-1}}\right]^{\frac{1}{p-1}} \left(\tfrac{q}{m+q}\right)^{\frac{q}{p-1}} \left[v^{\frac{m+q}{q}} - v_0^{\frac{m+q}{q}}\right]^{\frac{q}{p-1}} \\ &< C\left[v^{\frac{m+q}{q}} - v_0^{\frac{m+q}{q}}\right]^{\frac{q}{p-1}}, \end{aligned}$$

we get

$$v' < Cv^{\frac{m+q}{p-1}} < Cv\frac{m+1}{p-q}. \tag{2.23}$$

By using (2.23) we get

$$v < C(R-r)^{\frac{q-p}{(p-1)-(m+q)}}, \tag{2.24}$$

and

$$v' < C(R-r)^{\frac{m+1}{(p-1)-(m+q)}}. \tag{2.25}$$

While, the problem (1.2) reads as

$$\left(\phi_p(v')\right)' + \frac{N-1}{r}(v')^{p-1} = v^m (v')^q. \tag{2.26}$$

From (2.26) we find

$$(v')^{1-q}\left(\phi_p(v')\right)' + \frac{N-1}{r}(v')^{p-q} = v^m v', \tag{2.27}$$

integration for r we get

$$\int_{r_o}^{r}(v')^{1-q}\left(\phi_p(v')\right)' ds + \int_{r_0}^{r}\frac{N-1}{s}(v')^{p-q} = \int_{r_0}^{r} v^m v' ds,$$

$$(v')^{p-q} + \int_{r_0}^{r}\frac{N-1}{s}(v')^{p-q}ds = \frac{v^{m+1}-v_0^{m+1}}{m+1} + \int_{r_0}^{r}\phi_p(v')\left((v')^{1-q}\right)' ds,$$

$$(v')^{p-q} + (N-1)\int_{r_0}^{r}\frac{(v')^{p-q}}{s}ds > \frac{v^{m+1}-v_0^{m+1}}{m+1}. \tag{2.28}$$

Since $0 < q < 1$, by (2.27)

$$\int_{r_0}^{r}\frac{(v')^{p-q}}{s}ds < C(R-r)^{\frac{(m+2)(p-q)-(m+1)}{(p-1)-(m+q)}}.$$

From (2.28) we get

$$(v')^{p-q} > \frac{v^{m+1}-v_0^{m+1}}{m+1} - C(N-1)(R-r)^{\frac{(m+2)(p-q)-(m+1)}{(p-1)-(m+q)}},$$

$$(m+1)\frac{(v')^{p-q}}{v^{m+1}} > 1 - C(R-r),$$

$$(m+1)^{\frac{1}{p-1}}\frac{v'}{v^{\frac{m+1}{p-q}}} > 1 - C(R-r).$$

Integration for r we get

$$(m+1)^{\frac{1}{p-1}}\frac{q-p}{(m+q)-(p-1)}v^{\frac{(m+q)-(p-1)}{q-p}}\Bigg|_{r_0}^{r} > (R-r) - C(R-r)^2,$$

$$v < (m+1)^{\frac{1}{(m+q)-(p-1)}}\left[\frac{p-q}{(m+q)-(p-1)}\right]^{\frac{p-q}{(m+q)-(p-1)}}(R-r)^{\frac{q-p}{(m+q)-(p-1)}}\left[1-C(R-r)\right]^{\frac{q-p}{(m+q)-(p-1)}}.$$

Since

$$\left[1-C(R-r)\right]^{\frac{q-p}{(m+q)-(p-1)}} < 1 + C(R-r),$$

we get

$$v(r) < \phi_1(r)[1 + C(r-r)],$$

where $\phi_1(r)$ be the function defined by (2.18).

The theorem is proved.

3 Estimates for Boundary Blowup Solution

In this section we study the estimate for boundary blowup solution of problem (1.1) and (1.2).

Lemma 3.1. Let $\Omega \in R^N$, $N \geq 2$, be a bounded domain satisfying an interior and an exterior sphere condition at each point of its boundary $\partial\Omega$. Let ϕbe the function introduced in (2.2), let $u(x)$ be a solution to problem (1.1) in Ω, and let $\delta = \delta(x)$ be the distance from x to $\partial\Omega$. If $p > 1$, and $m + 1 > p - 1$ we have

$$\phi(\delta)(1 - C\delta) < u(x) < \phi(\delta)(1 + C\delta). \tag{3.1}$$

Proof. The proof uses theorem 2.1 and the comparison principle for elliptic equation(see for example $[15, Theorem 10.1]$).

Theorem 3.1. Let Ω be a bounded domain with a smooth boundary $\partial\Omega$, let ϕ be the function introduced in (2.2), and let $\delta = \delta(x)$ be the distance from x to $\partial\Omega$. Let $p > 1$, and $m + 1 > p - 1$. Define

$$w(x) = \phi(\delta)\left(1 + \frac{(p-1)(N-1)H(x)}{2((m+2)(p-1)-(m+1))}\delta + \alpha\delta^{\sigma}\right), \tag{3.2}$$

where $H(x)$ denotes the mean curvature of the surface ($\delta(x) = constant$) at the point x. If u is a solution to problem (1.1), $\sigma > 1$ is a suitable number and α is large enough then

$$u(x) \leq w(x).$$

Furthermore, if

$$v(x) = \phi(\delta)\left(1 + \frac{(p-1)(N-1)H(x)}{2((m+2)(p-1)-(m+1))}\delta - \alpha\delta^{\sigma}\right), \tag{3.3}$$

then

$$v(x) \leq u(x).$$

Proof. From (2.2) we find

$$\begin{aligned} \frac{\phi(t)}{-\phi'(t)} &= \frac{(m+1)-(p-1)}{p-1}t, \\ \frac{-\phi'(t)}{\phi''(t)} &= \frac{(m+1)-(p-1)}{m+1}t, \\ \frac{\phi(t)}{\phi''(t)} &= \frac{[(m+1)-(p-1)]^2}{(p-1)(m+1)}t^2. \end{aligned} \tag{3.4}$$

Let $K = (N-1)H$ and

$$A = \frac{(p-1)K}{2\left((m+2)(p-4)-(m+1)\right)}. \tag{3.5}$$

Then

$$w = \phi(\delta)(1 + A\delta + \alpha\delta^{\sigma}). \tag{3.6}$$

We have

$$\nabla w = \phi'\nabla w(1 + A\delta + \alpha\delta^{\sigma}) + \phi(\nabla A\delta + A\nabla\delta + \alpha\sigma\delta^{\sigma-1}\nabla\delta). \tag{3.7}$$

Since (see for example [10])

$$|\nabla\delta| = 1, \ \ \Delta\delta = -(N-1)H = -K,$$

we find

$$\begin{aligned} \Delta w \ &= (\phi''\nabla\delta\nabla\delta + \phi'\Delta\delta)(1 + A\delta + \alpha\delta^{\sigma}) + \phi'\nabla\delta(\nabla A\delta + A\nabla\delta + \alpha\sigma\delta^{\sigma-1}\nabla\delta) \\ &\quad + \phi'\nabla\delta(\nabla A\delta + A\nabla\delta + \alpha\sigma\delta^{\sigma-1}\nabla\delta) \\ &\quad + \phi\left(\Delta A\delta + \nabla A\nabla\delta + \nabla A\nabla\delta + a\Delta\delta + \alpha\sigma(\sigma-1)\delta^{\sigma-2}\nabla\delta + \alpha\sigma\delta^{\sigma-1}\Delta\delta\right) \\ &= (\phi'' - \phi'K)(1 + A\delta + \alpha\delta^{\sigma}) + 2\phi'(\nabla A\nabla\delta\delta + A + \alpha\sigma\delta^{\sigma-1}) \\ &\quad + \phi\left(\Delta A\delta + 2\nabla A\nabla\delta - AK + \alpha\sigma(\sigma-1)\delta^{\sigma-2} - \alpha\sigma\delta^{\sigma-1}K\right). \end{aligned}$$

By using (3.4) we find

$$\Delta w = \phi'' \Big[\Big(1 + \tfrac{(m+1)-(p-1)}{m+1} \delta K\Big) (1 + A\delta + \alpha\delta^\sigma) - 2 \tfrac{(m+1)-(p-1)}{m+1} \delta(\nabla A \nabla\delta\delta + A + \alpha\sigma\delta^{\sigma-1}) + \tfrac{[(m+1)-(p-1)]^2}{(p-1)(m+1)} \delta^2 \left(\Delta A\delta + 2\nabla A\nabla\delta - AK + \alpha\sigma(\sigma-1)\delta^{\sigma-2} - \alpha\sigma\delta^\sigma K\right)\Big],$$

we get

$$\Delta w = \phi'' \Big[1 + A\delta + \tfrac{(m+1)-(p-1)}{m+1} \delta(K - 2A) + O(1)\delta^2 + \alpha\delta^\sigma \Big(1 - 2\sigma \tfrac{(m+1)-(p-1)}{m+1} + \sigma(\sigma-1) \tfrac{[(m+1)-(p-1)]^2}{(p-1)(m+1)} + O(1)\delta\Big)\Big], \tag{3.8}$$

where $O(1)$ denotes a bounded quantity as $\delta \to 0$.

Now we estimate $|\nabla w|$.

$$\begin{aligned} \nabla w &= \phi' \nabla w(1 + A\delta + \alpha\delta^\sigma) + \phi(\nabla A\delta + A\nabla\delta + \alpha\sigma\delta^{\sigma-1}\nabla\delta) \\ &= \phi' \Big[\nabla\delta(1 + A\delta + \alpha\delta^\sigma) - \tfrac{(m+1)-(p-1)}{p-1} \delta(\nabla A\delta + A\nabla\delta + \alpha\sigma\delta^{\sigma-1}\nabla\delta)\Big] \\ &= \phi' \Big[\nabla\delta \Big(1 + A \tfrac{2(p-1)-(m+1)}{p-1} \delta + \alpha\delta^\sigma(1 - \tfrac{(m+1)-(p-1)}{p-1} \sigma)\Big) - \tfrac{(m+1)-(p-1)}{p-1} \nabla A\delta^2\Big]. \end{aligned}$$

Fix α and σ, we take δ so small that

$$1 + A \frac{2(p-1) - (m+1)}{p-1} \delta + \alpha\delta^\sigma (1 - \frac{(m+1) - (p-1)}{p-1} \sigma) > 0.$$

Then, we have

$$|\nabla w| = (-\phi') \left[1 + A \frac{2(p-1) - (m+1)}{p-1} \delta + \alpha\delta^\sigma (1 - \frac{(m+1) - (p-1)}{p-1} \sigma) + O(1)\delta^2\right]. \tag{3.9}$$

and

$$\begin{aligned} |\nabla w|^{p-2} &= (-\phi')^{p-2} \Big[1 + A \tfrac{2(p-1)-(m+1)}{p-1} \delta + \alpha\delta^\sigma (1 - \tfrac{(m+1)-(p-1)}{p-1} \sigma) + O(1)\delta^2\Big]^{p-2} \\ &= (-\phi')^{p-2} [1 + A \tfrac{2(p-1)-(m+1)}{p-1} \delta + \alpha\delta^\sigma (1 - \tfrac{(m+1)-(p-1)}{p-1} \sigma) \\ &\quad + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2], \end{aligned}$$

By using (3.8) we get

$$\begin{aligned} |\nabla w|^{p-2} \Delta w = (-\phi')^{p-2} \phi'' &\Big[1 + A\delta + \tfrac{(m+1)-(p-1)}{m+1} \delta(K - 2A) + A(p-2) \tfrac{2(p-1)-(m+1)}{p-1} \delta + O(1)\delta^2\Big] \\ + (-\phi')^{p-2} \phi'' (\alpha\delta^\sigma) &\Big[1 - 2\sigma \tfrac{(m+1)-(p-1)}{m+1} + \sigma(\sigma-1) \tfrac{[(m+1)-(p-1)]^2}{(p-1)(m+1)} \\ &+ (p-2)(1 - \tfrac{(m+1)-(p-1)}{p-1} \sigma) + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big]. \end{aligned} \tag{3.10}$$

Let us estimate w^m. We have

$$\begin{aligned} w^m &= \phi^m (1 + A\delta + \alpha\delta^\sigma)^m \\ &= \phi^m \Big(1 + mA\delta + m\alpha\delta^\sigma + m(m+1)(1+\omega)^{m+2} \tfrac{(A\delta + \alpha\delta^\sigma)^2}{2}\Big). \end{aligned} \tag{3.11}$$

Where ω is a quantity in between 0 and $A\delta + \alpha\delta^\sigma$. From now on, we choose α, σ and ρ such that

$$-\frac{1}{2} \le A\delta + \alpha\delta^\sigma \le 1.$$

Then $\frac{1}{2} < 1 + \omega < 2$, and

$$w^m = \phi^m \left(1 + mA\delta + m\alpha\delta^\sigma + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\right).$$

Since $\phi''(-\phi')^{p-2} = \phi^m(-\phi')$, by (3.9) and (3.11) we find

$$w^m|\nabla w| = \phi''(-\phi')^{p-2}\Big[1 + A\Big(m + \tfrac{2(p-1)-(m+1)}{p-1}\Big)\delta + \alpha\delta^\sigma\Big(m+1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma\Big) + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big] \tag{3.12}$$

Using (3.10) and (3.12), the inequality

$$div\left(|\nabla w|^{p-2}\nabla w\right) < w^m|\nabla w|$$

reads as

$$\begin{aligned}(-\phi')^{p-2}\phi''&\Big[1 + A\delta + \tfrac{(m+1)-(p-1)}{m+1}\delta(K-2A) + A(p-2)\tfrac{2(p-1)-(m+1)}{p-1}\delta + O(1)\delta^2\Big]\\ &+(-\phi')^{p-2}\phi''(\alpha\delta^\sigma)\Big(1 - 2\sigma\tfrac{(m+1)-(p-1)}{m+1} + \sigma(\sigma-1)\tfrac{[(m+1)-(p-1)]^2}{(p-1)(m+1)}\\ &+(p-2)(1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma) + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big)\\ &< \phi''(-\phi')^{p-2}\Big[1 + A\Big(m + \tfrac{2(p-1)-(m+1)}{p-1}\Big)\delta + \alpha\delta^\sigma\Big(m+1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma\Big)\\ &+O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big].\end{aligned} \tag{3.13}$$

We claim that

$$A + \frac{(m+1)-(p-1)}{m+1}(K-2A) + A(p-2)\frac{2(p-1)-(m+1)}{p-1}$$

$$= A\left(m + \frac{2(p-1)-(m+1)}{p-1}\right).$$

Indeed, we have

$$\begin{aligned}\tfrac{(m+1)-(p-1)}{m+1}(K-2A) &= A\tfrac{(m+1)(p-2)}{p-1} - A\tfrac{(p-2)[2(p-1)-(m+1)]}{p-1}\\ &= 2A\tfrac{p-2}{p-1}\left((m+1)-(p-1)\right),\end{aligned}$$

then we get

$$K - 2A = 2A\frac{(m+1)(p-2)}{p-1},$$

and

$$K = 2A\frac{(m+2)(p-1)-(m+1)}{p-1}.$$

The latter equation follows easily from (3.5). Hence, inequality (3.13) holds provided

$$\begin{aligned}C_1\delta^2 + \alpha\delta^\sigma\Big(1 - 2\sigma\tfrac{(m+1)-(p-1)}{m+1} + \sigma(\sigma-1)\tfrac{[(m+1)-(p-1)]^2}{(p-1)(m+1)} + (p-2)\Big(1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma\Big)\Big)\\ < \alpha\delta^\sigma\Big(m+1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma + C_2\delta + C_3\alpha\delta^\sigma\Big),\end{aligned}$$

where C_1, C_2 and C_3 are suitable constant. After simplification we find

$$\begin{aligned}C_1\delta^2 \le \alpha\delta^\sigma\left((m+1)-(p-1)\right)\Big(1 - \tfrac{p-3}{p-1}\sigma + \tfrac{2}{m+1}\sigma\\ -\sigma(\sigma-1)\tfrac{(m+1)-(p-1)}{(p-1)(m+1)} - C_2\delta + C_3\alpha\delta^\sigma\Big).\end{aligned} \tag{3.14}$$

The quantity

$$1 - \frac{p-3}{p-1}\sigma + \frac{2}{m+1}\sigma - \sigma(\sigma-1)\frac{(m+1)-(p-1)}{(p-1)(m+1)},$$

computed at $\sigma = 1$ becomes

$$2\frac{(m+1)+(p-1)}{(m+1)(p-1)}.$$

Which is positive. By continuity, we have

$$1-\frac{p-3}{p-1}\sigma+\frac{2}{m+1}\sigma-\sigma(\sigma-1)\frac{(m+1)-(p-1)}{(p-1)(m+1)}>0,$$

with a suitable $\sigma>1$. Fixed such a value of σ, choose α and δ so that

$$1-\frac{p-3}{p-1}\sigma+\frac{2}{m+1}\sigma-\sigma(\sigma-1)\frac{(m+1)-(p-1)}{(p-1)(m+1)}-C_2\delta+C_3\alpha\delta^\sigma>0.$$

The inequality (3.13) (and the inequality $div\left(|\nabla w|^{p-2}\nabla w\right)<w^m|\nabla w|$) holds for α large enough and x such that $\delta(x)\le\delta_0$, with a suitable δ_0.

Consider the domain $\Omega_{\delta_1}=\{x\in\Omega,\ \ \delta(x)<\delta_0\}$. Let us show that, for δ_1 small enough, $u(x)\le w(x)$ on Ω_{δ_1}. Indeed, by lemma 3.1, we know that

$$w(x)<\phi(\delta)(1+C\delta).$$

Hence,

$$\begin{aligned}w(x)-u(x)\ &>\phi(\delta)(1+A\delta+\alpha\delta^\sigma)-\phi(\delta)(1+C\delta)\\ &=\phi(\delta)\left((A-C)\delta+\alpha\delta^\sigma\right).\end{aligned}$$

Let α_0 and δ_0 such the inequality (3.13) holds for $\delta\le\delta_0$. Decrease δ(increasing α so that $\alpha_1\delta_1^\sigma=\alpha_0\delta_0^\sigma$) until

$$(A-C)\delta_1+\alpha_1\delta_1^\sigma>0.$$

Then $w(x)\ge u(x)$ for $\delta(x)=\delta_1$.

Now we introduce a number $0<\theta<1$, of course, we have $w(x)>\theta u(x)$ for x such that $\delta(x)=\delta_1$. On the other hand, using lemma 3.1 again we have

$$w(x)-\theta u(x)>\phi(\delta)\left(1-\theta+(A-C\theta)\delta+\alpha\delta^\sigma\right).$$

As $\delta\to0$ (with α fixed) we have

$$1-\theta+(A-C\theta)\delta+\alpha\delta^\sigma>0.$$

Hence, $w(x)-\theta u(x)>0$ near $\partial\Omega$.

Since $0<\theta<1$ and $m+1-(p-1)>0$, by (1.1) we find

$$\operatorname{div}\left(|\nabla(\theta u)|^{p-2}\nabla(\theta u)\right)>(\theta u)^m|\nabla(\theta u)| \tag{3.15}$$

Indeed, since

$$\Delta_p(u)=u^m|\nabla u|,$$

we find

$$\Delta_p(\theta u)=\theta^{p-1}\Delta_p u,$$

and

$$(\theta u)^m|\nabla(\theta u)|=\theta^{m+1}u^m|\nabla u|,$$

then we get

$$\Delta_p(\theta u)/(\theta u)^m|\nabla(\theta u)|=\theta^{p-1-(m+1)}>1.$$

The (3.15), together with the inequality $\operatorname{div}\left(|\nabla w|^{p-2}\nabla w\right)<w^m|\nabla w|$, and the condition $\theta u(x)\le w(x)$ on $\partial\Omega_{\delta_1}$, imply that $\theta u(x)\le w(x)$ on Ω_{δ_1}. As $\theta\to1$, we find $u(x)\le w(x)$ on Ω_{δ_1}. Increasing α we get $u(x)\le w(x)$ on Ω. The first assertion of the theorem follows.

To get the inequality $v(x) \le u(x)$. We adopt a similar argument. to place of (3.10) we find, with $v = \phi(\delta)(1 + A\delta - \alpha\delta^\sigma)$, where A is as in (3.5),

$$\begin{aligned}|\nabla v|^{p-2}\Delta v \ &= (-\phi')^{p-2}\phi''[1 + A\delta + \tfrac{(m+1)-(p-1)}{m+1}\delta(K-2A) + A(p-2)\tfrac{2(p-1)-(m+1)}{p-1}\delta \\ &+O(1)\delta^2] - (-\phi')^{p-2}\phi''(\alpha\delta^\sigma)(1 - 2\sigma\tfrac{(m+1)-(p-1)}{m+1} + \sigma(\sigma-1)\tfrac{[(m+1)-(p-1)]^2}{(p-1)(m+1)} \\ &+(p-2)(1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma) + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2).\end{aligned} \tag{3.16}$$

In place of (3.12), we have

$$\begin{aligned}v^m|\nabla v| = \phi''(-\phi')^{p-2}\Big[1 + A\Big(m + \tfrac{2(p-1)-(m+1)}{p-1}\Big)\delta - \alpha\delta^\sigma\Big(m + 1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma\Big) \\ +O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big].\end{aligned} \tag{3.17}$$

Using (3.16) and (3.17), the inequality

$$\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) > v^m|\nabla v| \tag{3.18}$$

reads as

$$\begin{aligned}&(-\phi')^{p-2}\phi''\Big[1 + A\delta + \tfrac{(m+1)-(p-1)}{m+1}\delta(K-2A) + A(p-2)\tfrac{2(p-1)-(m+1)}{p-1}\delta + O(1)\delta^2\Big] \\ &\quad -(-\phi')^{p-2}\phi''(\alpha\delta^\sigma)\Big(1 - 2\sigma\tfrac{(m+1)-(p-1)}{m+1} + \sigma(\sigma-1)\tfrac{[(m+1)-(p-1)]^2}{(p-1)(m+1)} \\ &\quad +(p-2)(1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma) + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big) \\ &\quad > \phi''(-\phi')^{p-2}\Big[1 + A\Big(m + \tfrac{2(p-1)-(m+1)}{p-1}\Big)\delta - \alpha\delta^\sigma\Big(m + 1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma\Big) \\ &\quad +O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big].\end{aligned} \tag{3.19}$$

After simplification we find

$$\begin{aligned}-C_1\delta^2 - \alpha\delta^\sigma(1 - 2\sigma\tfrac{(m+1)-(p-1)}{m+1} + \sigma(\sigma-1)\tfrac{[(m+1)-(p-1)]^2}{(p-1)(m+1)} + (p-2)(1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma)) \\ > -\alpha\delta^\sigma\Big(m + 1 - \tfrac{(m+1)-(p-1)}{p-1}\sigma + C_2\delta - C_3\alpha\delta^\sigma\Big),\end{aligned} \tag{3.20}$$

which is equivalent to (3.14). Hence, we have $div\left(|\nabla v|^{p-2}\nabla v\right) > v^m|\nabla v|$ for large enough and x such that $\delta(x) \le \delta_0$, $u(x) \ge v(x)$ on Ω_{δ_1}. Indeed, by lemma 3.1 we know that

$$u(x) > \phi(\delta)(1 - C\delta).$$

Hence,

$$v(x) - u(x) < \phi(\delta)((A + C)\delta - \alpha\delta^\sigma).$$

Let α_0 and δ_0 such that inequality (3.20) holds for $\delta \le \delta_0$. Decrease δ (increasing α so that $\alpha_1\delta_1^\sigma = \alpha_0\delta_0^\sigma$) until

$$(A + C)\delta_1 - \alpha_1\delta_1^\sigma < 0.$$

Then $u(x) \ge v(x)$ for $\delta(x) = \delta_1$.

Now, for $\Theta > 1$ we have $v(x) < \Theta u(x)$ for x such that $\delta(x) > \delta_1$. On the other hand, by lemma 2.2 it follows that $v(x) \le \Theta u(x)$ for x near $\partial\Omega$. We have proved that proved that $v(x) \le \Theta u(x)$ on $\partial\Omega_{\delta_1}$. Since $\Theta > 1$ and $m + 1 - (p-1) > 0$, by (1.1a) we find

$$\Delta_p(\Theta u) < (\Theta u)^m|\nabla(\Theta u)|.$$

The latter inequality, together with the inequality (3.18) and the condition $v(x) \le \Theta u(x)$ on $\partial\Omega_{\delta_1}$, imply that $v(x) \le \Theta u(x)$ on Ω_{δ_1}. As $\Theta \to 1$ we find $v(x) \le u(x)$ on Ω_{δ_1}. Increasing α we get $v(x) \le u(x)$ on Ω.

The theorem is proved.

Now, when $p > \ 0 < q < 1$, and $m + q > p - 1$, we get partial argument similar to Theorem 3.1.

Lemma 3.2. Similar to lemma 3.1, ϕ_1 be the function introduced in (2.18), let $u(x)$ be a solution to problem (1.1a) in Ω. If $p > \ 0 < q < 1$, and $m + q > p - 1$, we have

$$u(x) < \phi(\delta)(1 + C\delta).$$

Theorem 3.2. Let Ω be a bounded domain with a smooth boundary $\partial\Omega$, let ϕ be the function introduced in (2.18), and let $\delta = \delta(x)$ be the distance from x to $\partial\Omega$. Let $p > 1, \ \ 0 < q < 1$, and $m + 1 > p - 1$. Define

$$w(x) = \phi(\delta)\left(1 + \frac{(p-q)(N-1)H(x)}{2((m+2)(p-q)-(m+1))}\delta + \alpha\delta^\sigma\right),$$

where $H(x)$ denotes the mean curvature of the surface ($\delta(x) = \textit{constant}$) at the point x. If u is a solution to problem (1.2), $\sigma > 1$ is a suitable number and α is large enough then

$$u(x) \le w(x).$$

Proof. From (2.18) we find

$$\begin{aligned}\frac{\phi(t)}{-\phi'(t)} &= \frac{(m+q)-(p-1)}{p-q}t,\\ \frac{-\phi'(t)}{\phi''(t)} &= \frac{(m+q)-(p-1)}{m+1}t,\\ \frac{\phi(t)}{\phi''(t)} &= \frac{[(m+q)-(p-1)]^2}{(p-q)(m+1)}t^2.\end{aligned} \tag{3.21}$$

Let $K = (N-1)H$ and

$$A = \frac{(p-q)K}{2\left((m+2)(p-q)-(m+1)\right)}, \tag{3.22}$$

then

$$w = \phi(\delta)(1 + A\delta + \alpha\delta^\sigma).$$

In place of (3.8) we have

$$\begin{aligned}\Delta w = \phi''\Big[&1 + A\delta + \tfrac{(m+q)-(p-1)}{m+1}\delta(K-2A) + O(1)\delta^2\\ &+\alpha\delta^\sigma\left(1 - 2\sigma\tfrac{(m+q)-(p-1)}{m+1} + \sigma(\sigma-1)\tfrac{[(m+q)-(p-1)]^2}{(p-q)(m+1)} + O(1)\delta\right)\Big].\end{aligned} \tag{3.23}$$

Then we get the estimate for $|\nabla w|$,

$$|\nabla w| = (-\phi')[1 + A\tfrac{(p-q)+(p-1)-(m+q)}{p-q}\delta + \alpha\delta^\sigma\left(1 - \tfrac{(m+q)-(p-1)}{p-q}\sigma\right) + O(1)\delta^2]. \tag{3.24}$$

In place of (3.10) we get

$$\begin{aligned}|\nabla w|^{p-2}\Delta w = (-\phi')^{p-2}\phi''\Big[&1 + A\delta + \tfrac{(m+q)-(p-1)}{m+1}\delta(K-2A)\\ &+A(p-2)\tfrac{(p-q)+(p-1)-(m+q)}{p-1}\delta + O(1)\delta^2\Big]\\ &+(-\phi')^{p-2}\phi''(\alpha\delta^\sigma)\Big[1 - 2\sigma\tfrac{(m+q)-(p-1)}{m+1} + \sigma(\sigma-1)\tfrac{[(m+q)-(p-1)]^2}{(p-q)(m+1)}\\ &+(p-2)(1 - \tfrac{(m+q)-(p-1)}{p-1}\sigma) + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big].\end{aligned} \tag{3.25}$$

Let us estimate $|\nabla w|^q$. By using (3.9) we get

$$\begin{aligned}|\nabla w|^q &= (-\phi')^q\left[1 + A\tfrac{(p-q)+(p-1)-(m+q)}{p-q}\delta + \alpha\delta^\sigma\left(1 - \tfrac{(m+q)-(p-1)}{p-q}\sigma\right) + O(1)\delta^2\right]^q\\ &= (-\phi')^q[qA\tfrac{(p-q)+(p-1)-(m+q)}{p-q}\delta + q\alpha\delta^\sigma\left(1 - \tfrac{(m+q)-(p-1)}{p-q}\sigma\right) + O(1)\delta^2+\\ &O(1)(\alpha\delta^\sigma)^2].\end{aligned} \tag{3.26}$$

By using (3.11) and (3.26) we have

$$\begin{aligned} w^m|\nabla w|^q = \phi^m(-\phi')^q \Big[1 + A\left(m + q\tfrac{(p-q)+(p-1)-(m+q)}{p-q}\right) \\ +\alpha\delta^\sigma\left(m+q-q\tfrac{(m+q)-(p-1)}{p-q}\sigma\right) + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big]. \end{aligned} \tag{3.27}$$

By (3.25) and (3.27), the inequality

$$\operatorname{div}(|\nabla w|^{p-2}\nabla w) < w^p|\nabla w|^q$$

reads as

$$\begin{aligned} &(-\phi')^{p-2}\phi''\left[1 + A\delta + \tfrac{(m+q)-(p-1)}{m+1}\delta(K-2A) + A(p-2)\tfrac{(p-q)+(p-1)-(m+q)}{p-q}\delta + O(1)\delta^2\right] \\ &\quad +(-\phi')^{p-2}\phi''(\alpha\delta^\sigma)\Big(1 - 2\sigma\tfrac{(m+q)-(p-1)}{m+1} + \sigma(\sigma-1)\tfrac{[(m+q)-(p-1)]^2}{(p-q)(m+1)} \\ &\quad +(p-2)(1-\tfrac{(m+q)-(p-1)}{p-1}\sigma) + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big) \\ &\quad < \phi''(-\phi')^{p-2}\Big[1 + A\left(m + q\tfrac{(p-q)+(p-1)-(m+q)}{p-1}\right)\delta \\ &\quad +\alpha\delta^\sigma\left(m+q-q\tfrac{(m+q)-(p-1)}{p-q}\sigma\right) + O(1)\delta^2 + O(1)(\alpha\delta^\sigma)^2\Big]. \end{aligned} \tag{3.28}$$

We claim that

$$A + \tfrac{(m+q)-(p-1)}{m+1}(K-2A) + A(p-2)\tfrac{(p-q)+(p-1)-(m+q)}{p-q} = A\left(m + q\tfrac{(p-q)+(p-1)-(m+q)}{p-1}\right).$$

Indeed, we have

$$\frac{(m+q)-(p-1)}{m+1}(K-2A) = \frac{2(p-q-1)(m+q)-(p-1)}{p-q},$$

$$K = 2A + 2\frac{(m+1)(p-q-1)}{p-q},$$

$$K = 2\frac{(m+2)(p-q)-(m+1)}{p-q}.$$

The latter equation follows easily from (3.22). Hence, (3.28) holds provided

$$\begin{aligned} C_1\delta^2 + \alpha\delta^\sigma\left(1 - 2\sigma\tfrac{(m+q)-(p-1)}{m+1} + \sigma(\sigma-1)\tfrac{[(m+q)-(p-1)]^2}{(p-1)(m+q)} + (p-2)\left(1 - \tfrac{(m+q)-(p-1)}{p-1}\sigma\right)\right) \\ < \alpha\delta^\sigma\left(m+q-q\tfrac{(m+q)-(p-1)}{p-q}\sigma - C_2\delta + C_3\alpha\delta^\sigma\right), \end{aligned}$$

where C_1, C_2, and C_3 are suitable constants. After simplification we find

$$\begin{aligned} C_1\delta^2 \le \alpha\delta^\sigma\left((m+q)-(p-1)\right)\Big(1 - \tfrac{2(p-1)}{p-q}\sigma \\ +\tfrac{2}{m+1}\sigma - \sigma(\sigma-1)\tfrac{(m+q)-(p-1)}{(p-1)(m+q)} - C_2\delta + C_3\alpha\delta^\sigma\Big). \end{aligned} \tag{3.29}$$

Which is equivalent to (3.14). Hence, we have

$$\operatorname{div}(|\nabla w|^{p-2}\nabla w) < w^p|\nabla w|^q$$

for a large enough and x such that $\delta(x) \le \delta_0$, with a suitable δ_0. Arguing as in the proof of the previous theorem one prove that $w(x) \ge u(x)$ in Ω.

4 Conclusion

We introduce the concept of the boundary blowup solutions of p-Laplacian type quasilinear elliptic equations. We obtain that the estimate of the radial solution in the annulus, and that the estimate of the boundary blowup solution on a bounded domain.

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Competing Interests

The authors declare that no competing interests exist.

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