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Hilbert's Fourth Problem as a Possible Candidate on the MILLENNIUM PROBLEM in Geometry

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Abstract

Hilbert's Fourth Problem is one of the most important mathematical problems, formulated by Hilbert in 1900. Unfortunately, attempts to solve this problem during 20th century did not lead to the generally recognized solution, and now modern mathematicians believe that the problem has been formulated by Hilbert "very vague" and therefore it can not be solved. The main purpose of this article is to develop a new view on authors' original solution to this problem and to interpret this problem as MILLENNIUM PROBLEM in Geometry what has an interdisciplinary importance and affects not only on geometry, but also on all theoretical natural sciences. The source of a new approach to solving this problem is a new branch of mathematics, the Mathematics of Harmony, which goes back in its origins to Euclid's Elements and has interdisciplinary importance for modern science.

Keywords: Millennium problems; Hilbert's fourth problem; the golden ratio; Fibonacci and Lucas numbers; Binet's formulas; Bodnar's geometry; Gazale's formulas; recursive hyperbolic functions; original solution to Hilbert's fourth problem.

1 Introduction

In the recent years, the so-called *Millennium Problems* became a big dragging of mathematicians and physicists. The outstanding mathematician **David Hilbert** gave the beginning of this dragging. In 1900 he

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presented twenty-three Great Mathematical Problems at the International Congress of Mathematicians in Paris. Explaining the purpose of the formulation of his "Mathematical Problems," **David Hilbert** writes [1]:

> *"For the close of a great epoch not only invites us to look back into the past but also directs our thoughts to the unknown future".*

Thus, Hilbert invites us not only to look to the past, but also to direct our efforts into the unknown future.

As outlined in the article [2], *"Hilbert's address of 1900 to the International Congress of Mathematicians in Paris is perhaps the most influential speech ever given to mathematicians, given by a mathematician, or given about mathematics. In it, Hilbert outlined 23 major mathematical problems to be studied in the coming century… Hilbert's address was more than a collection of problems. It outlined his philosophy of mathematics and proposed problems important to his philosophy".*

Modern mathematicians decided to continue the great tradition of David Hilbert. In May 2000, by emulating to Hilbert, the *Clay Mathematics Institute of Cambridge* announced (in Paris, for full effect) about seven "Millennium Prize Problems," each with a bounty of $1 million [3].

Modern physicists have decided not lag from mathematicians. They have formulated 10 Physics Problems for the Next Millennium [4]. These physical problems are striking our imagination and therefore are called "Millennium Madness".

The analysis of the list of the "Millennium Prize Problems" [3], leaves some dissatisfaction. For example, only one unsolved mathematical problem from Hilbert's list of 23 Mathematical Problems has been included into the list of Clay Mathematics Institute Millennium Prize Problems. We are talking about the **Riemann hypothesis,** which is well-known as **Hilbert's Eighth Problem** [5].

Of course, the authors do not have intention to doubt in the Riemann hypothesis, which has fundamental interest. However, it is surprising that Hilbert's some unsolved mathematical problems, which have the same fundamental interest, are not included into the list of Millennium Problems, compiled by Clay Mathematics Institute.

The authors simply would like to show that in Hilbert's list there are important mathematical problems, which deserve to be called the MILLENIUM PROBLEMS. As an example, the authors chose **Hilbert's Fourth Problem**, which concerns hyperbolic geometry and have interdisciplinary significance for many branches of mathematics and theoretical natural sciences.

Hilbert's Fourth Problem is considered by modern mathematical community as unsolved. Wikipedia [5] has reflected the opinion of modern mathematical community on the solution of this problem as follows: *"Too vague to be stated resolved or not."* This means that the modern mathematical community has placed all responsibility for solution (or rather, for the lack of solution) of this problem on Hilbert himself, who formulated this problem *too vague*.

Our studies of this problem are set out in the publications [6-10]. The purpose of this article is to present the original solution to Hilbert's Fourth Problem [6-10] in a popular form, accessible to all mathematicians, teachers and students of mathematics, as well as representatives of theoretical natural sciences, interested in new classes of non-Euclidean geometries.

2 Hilbert's Fourth Problem

2.1 A Little of History

In the lecture *Mathematical Problems* [1], presented at the Second International Congress of Mathematicians (Paris, 1900), the prominent mathematician **David Hilbert** (1862-1943) formulated his famous 23 mathematical problems. These problems determined considerably the development of mathematics in the 20^{th} century. This lecture is a unique event in mathematics history and in mathematical literature.

In [1], this problem has been formulated as follows:

> ***"The more general question now arises: Whether from other suggestive standpoints geometries may not be devised which, with equal right, stand next to Euclidean geometry".***

Hilbert's quote contains the formulation of very important mathematical problem, which touches, according to Hilbert, to the foundation of *geometry, number theory, theory of surfaces* and *calculus.* Hilbert's Fourth Problem is of fundamental interest not only for mathematics, but also for all theoretical natural sciences:

> ***"Whether exist non-Euclidean geometries, which are close to Euclidean geometry and are interesting from the "other suggestive standpoints"?"***

If we consider this problem in the context of theoretical natural sciences, then the goal of Hilbert's Fourth Problem is to search NEW HYPERBOLIC WORLDS OF NATURE, which are close to Euclidean geometry and reflect some new properties of Nature's structures and phenomena.

Hilbert considers *Lobachevski's geometry* and *Riemannian geometry* as the nearest to Euclidean geometry. As it is noted in Wikipedia [11], *"in mathematics, Hilbert's Fourth Problem in the 1900 "Hilbert problems" was a foundational question in geometry. In one statement derived from the original, it was to find geometries whose axioms are closest to those of Euclidean geometry if the ordering and incidence axioms are retained, the congruence axioms weakened, and the equivalent of the parallel postulate omitted".*

As follows from the Introduction to the "Mathematical Problems" [1], Hilbert pays special attention to this problem, emphasizing its **interdisciplinary nature**.

The fact, that mathematicians for a century not been able to solve Hilbert's Fourth Problem, highlights the **complexity of the problem** and its **undoubted importance for mathematics and theoretical science.**

It is clear that **we cannot ignore this really outstanding mathematical problem, formulated by Hilbert in 1900.**

2.2 Critical Analysis of the Known Attempts to Solve Hilbert's Fourth Problem

In mathematical literature Hilbert's Fourth Problem is sometimes considered as formulated *very vague* what makes difficult its final solution. As it is noted in Wikipedia article [11], *"the original statement of Hilbert, however, has also been judged too vague to admit a definitive answer".*

Unfortunately, the attempts to resolve Hilbert's Fourth Problem, made by German mathematician **Herbert Hamel** (1901) and later by the Soviet mathematician **Alexey Pogorelov** [12] have not led to significant progress, as follows from Wikipedia**.** As mentioned above, in Wikipedia's articles [5,11], the status of the problem is determined as *"too vague*" and Pogorelov's book [12] even is not mentioned.

Similar point of view on Pogorelov's solution to Hilbert's Fourth Problem [12] is presented in the remarkable book [13]. Thus, from the standpoint of modern mathematical community, Hilbert's mistake was in the fact that he formulated this problem not clearly enough and this is the main reason, why Hilbert's Fourth Problem is not solved until now.

In spite of critical attitude of mathematicians to Hilbert's Fourth Problem, we should emphasize a great importance of this problem for mathematics and theoretical natural sciences. Without doubts, Hilbert's intuition led him to the conclusion that *Lobachevski's geometry* and *Riemannian geometry* did not exhaust all possible variants of non-Euclidean geometries. Hilbert's Fourth Problem directs researchers to searching of new non-Euclidean geometries, which are close to the traditional Euclidean geometry.

2.3 From the "Game of Postulates" to the "Game of Functions"

According to [14], the cause of the difficulties, arising at the solution of Hilbert's Fourth Problem, lies elsewhere. All the known attempts to resolve this problem (Herbert Hamel, Alexey Pogorelov) were based on the traditional approach and have been reduced to the so-called *"game of postulates"* [14].

This "game" in geometry started from the works by **Nikolai Lobachevski** and **Janos Bolyai**, when *Euclid's 5th postulate* was replaced on the opposite one. This was the most major step in the development of the *non-Euclidean geometry*, which led to *Lobachevski's geometry*. This geometry is considered as the most important mathematical discovery of the 19th century and rightly can be named the MILLENNIUM PROBLEM. It changed the traditional geometric ideas and led to the creation of *hyperbolic geometry*. It must be emphasized that the title of *hyperbolic geometry* highlights the fact that this geometry is based on the *hyperbolic functions*. The use of hyperbolic functions for mathematical description of Lobachevski's geometry is one of its "key" ideas.

2.4 New Approach to the Solution of Hilbert's Fourth Problem

It is important to emphasize one more that the very title of *hyperbolic geometry* contains in itself the important idea of another approach to the resolution of Hilbert's Fourth Problem. This idea consists in *searching new classes of hyperbolic functions*, which can be the basis for new hyperbolic geometries. Every new class of the hyperbolic functions "generates" new variant of hyperbolic geometry. By analogy with the *game of postulates* this approach to the solution of Hilbert's Fourth Problem can be named the *game of functions* [14].

3 New Class of the Recursive Hyperbolic Functions as the Way to New Hyperbolic Geometries

3.1 The "Extended" Fibonacci and Lucas Numbers

The Fibonacci and Lucas numbers F_n : 1,1,2,3,5,8,13,21,34,… and L_n: 1,3,4,7,11,18,29,47, …, given by the following recurrence relations:

$$F_n = F_{n-1} + F_{n-2}; \quad F_1 = F_2 = 1, \tag{1}$$

$$L_n = L_{n-1} + L_{n-2}; \quad L_1 = 1, L_2 = 3, \tag{2}$$

allow the following "extension" to the side of negative values of the index *n* (see Table 1).

Table 1. The "Extended" Fibonacci and Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10
F_n	0	1	1	2	3	5	8	13	21	34	55
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55
L_n	2	1	3	4	7	11	18	29	47	76	123
L_{-n}	2	-1	3	-4	7	-11	18	-29	47	-76	123

As follows from Table 1, the "extended" Fibonacci and Lucas numbers are connected with the following simple relations:

$$F_{-n}=(-1)^{n+1}F_n;\;\; L_{-n}=(-1)^n L_n \tag{3}$$

3.2 Cassini's Formula

There exists the following remarkable formula (*Cassini's formula*), which connects the adjacent Fibonacci numbers:

$$F_n^2-F_{n-1}F_{n+1}=(-1)^{n+1}. \tag{4}$$

Partial cases:

$$\begin{aligned}
&n=1:(1)^2-0\times1=(-1)^2=1;\\
&n=2:(1)^2-1\times2=(-1)^3=-1;\\
&n=3:(2)^2-1\times3=(-1)^4=+1;\\
&n=4:(3)^2-2\times5=(-1)^5=-1
\end{aligned}$$

3.3 Binet's Formulas

The "extended" Fibonacci and Lucas numbers (Table 1) can be represented explicitly through the "golden ratio" $\Phi=\frac{1+\sqrt{5}}{2}$ [15]:

$$F_n=\begin{cases}\dfrac{\Phi^n+\Phi^{-n}}{\sqrt{5}} \text{ for } n=2k+1\\[2mm] \dfrac{\Phi^n-\Phi^{-n}}{\sqrt{5}} \text{ for } n=2k\end{cases} \tag{5}$$

$$L_n=\begin{cases}\Phi^n+\Phi^{-n} \text{ for } n=2k\\ \Phi^n-\Phi^{-n} \text{ for } n=2k+1\end{cases} \tag{6}$$

The formulas (5), (6) are called *Binet's formulas.* These formulas were obtained by French mathematician **Binet** in 1843, although these formulas were known to **Euler, Daniel Bernoulli**, and **de Moivre** more than a century earlier. In particular, **de Moivre** obtained these formulas in 1718.

3.4 Recursive Hyperbolic Fibonacci and Lucas Functions

3.4.1 Classical hyperbolic functions

$$\textbf{Hypebolic sine}: sh(x)=\frac{e^x-e^{-x}}{2} \tag{7}$$

$$\textbf{Hypebolic cosine}: ch(x)=\frac{e^x+e^{-x}}{2} \tag{8}$$

$$\textbf{Analog of Phythagoras theorem}: ch^2x-sh^2x=1 \tag{9}$$

$$\textbf{Parity property}: sh(-x)=-sh(x);\quad ch(-x)=ch(x). \tag{10}$$

3.4.2 Hyperbolic Fibonacci and Lucas functions

Comparing Binet's formulas, represented in the forms (5), (6), to the classical hyperbolic functions (7), (8), we can see a similarity between them. This similarity caused the Ukrainian mathematicians **Alexey Stakhov** and **Ivan Tkachenko** to introduce the first version of the hyperbolic Fibonacci and Lucas functions, described in 1993 article [16]. The improved version of the hyperbolic Fibonacci and Lucas functions, have been introduced in **Stakhov** and **Rozin's** article [17], published in 2004.

Hyperbolic Fibonacci sine:

$$sF(x)=\frac{\Phi^x-\Phi^{-x}}{\sqrt{5}} \tag{11}$$

Hyperbolic Fibonacci cosine:

$$cF(x)=\frac{\Phi^x+\Phi^{-x}}{\sqrt{5}} \tag{12}$$

Hyperbolic Lucas sine:

$$sL(x)=\Phi^x-\Phi^{-x} \tag{13}$$

Hyperbolic Lucas cosine:

$$cL(x)=\Phi^x+\Phi^{-x} \tag{14}$$

3.5 The Graphs of the Hyperbolic Fibonacci and Lucas Functions

Comparing Binet's formulas (5), (6) to the hyperbolic Fibonacci and Lucas functions (11) - (14), it is easy to see that for the discrete values of the variable x (x=0,±1,±2,±3,…) the functions (11) - (14) are reduced to the "extended" Fibonacci and Lucas numbers calculated according to Binet's formula (5), (6), i.e.,

$$F_n=\begin{cases} sF(n) \text{ for } n=2k \\ cF(n) \text{ for } n=2k+1 \end{cases}; \tag{15}$$

$$L_n=\begin{cases} sL(n) \text{ for } n=2k+1 \\ cL(n) \text{ for } n=2k \end{cases}. \tag{16}$$

To demonstrate this property more clearly, we consider the graphs of the hyperbolic Fibonacci and Lucas functions, shown in Figs. 1 and 2.

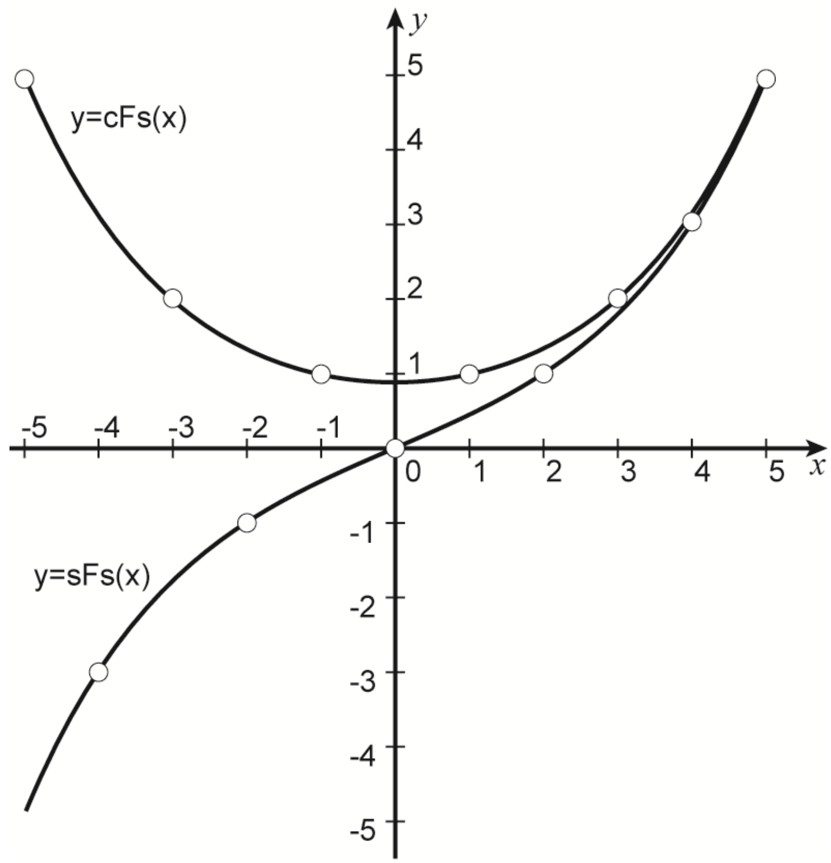

Fig. 1. The graphs of the hyperbolic Fibonacci functions

In Fig. 1, the graphs of the hyperbolic Fibonacci sine $y = sF(x)$ and the hyperbolic Fibonacci cosine $y = cF(x)$ are shown.

The points on the graph $y = sF(x)$ correspond to the "extended" Fibonacci numbers with the *even* indexes $2n$:

$$F_{2n} = \{..., F_{-8} = -21, F_{-6} = -8, F_{-4} = -3, F_{-2} = -1, F_0 = 0, F_2 = 1, F_4 = 3, F_6 = 8, F_8 = 21, ...\}. \quad (17)$$

The points on the graph $y = cF(x)$ correspond to the "extended" Fibonacci numbers with the *odd* indexes $2n+1$:

$$F_{2n+1} = \{..., F_{-7} = 13, F_{-5} = 5, F_{-3} = 2, F_{-1} = 1, F_1 = 1, F_3 = 3, F_5 = 5, F_7 = 13, ...\} \quad (18)$$

In Fig. 2, the graphs of the hyperbolic Lucas sine $y = sL(x)$ and the hyperbolic Lucas cosine $y = cL(x)$ are shown.

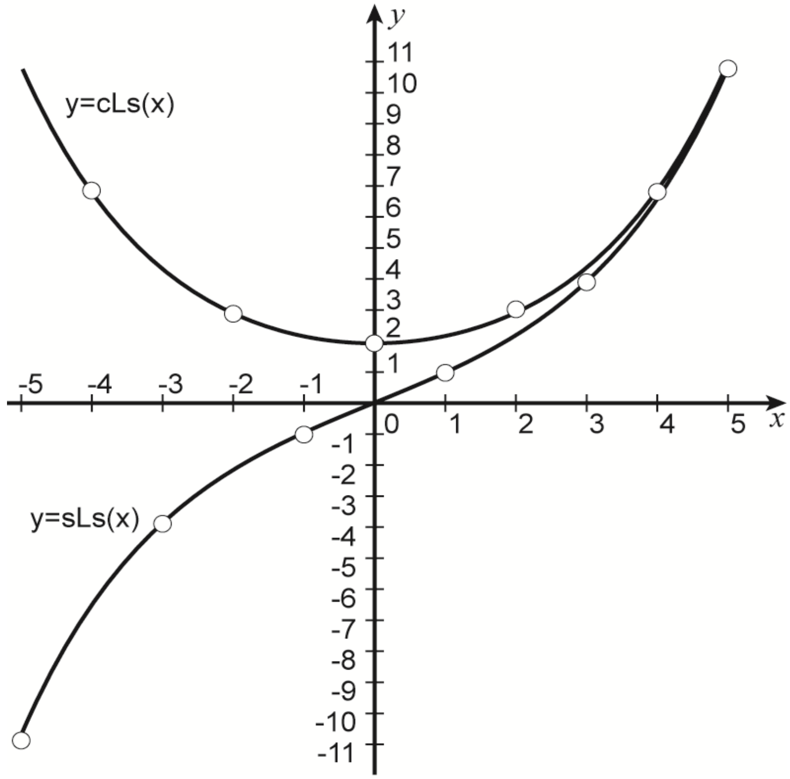

Fig. 2. The graphs of the hyperbolic Lucas functions

The points on the graph $y = sL(x)$ correspond to the "extended" Lucas numbers with the *odd* indexes $2n+1$:

$$L_{2n+1} = \{..., L_{-7} = -29, L_{-5} = -11, L_{-3} = -4, L_{-1} = -1, L_1 = 1, L_3 = 4, L_5 = 11, L_7 = 29, ...\} \quad (19)$$

The points on the graph $y = cL(x)$ correspond to the "extended" Lucas numbers with the *even* indexes $2n$:

$$L_{2n} = \{..., L_{-6} = 18, L_{-4} = 7, L_{-2} = 3, L_0 = 2, L_2 = 3, L_4 = 7, L_6 = 18, ...\} \quad (20)$$

Here it is necessary to point out that in the point x=0 the hyperbolic Fibonacci cosine $cF(x)$ takes the value $cF(0) = \frac{2}{\sqrt{5}}$ (Fig. 1), and the hyperbolic Lucas cosine $cL(x)$ takes the value $cL(0)$=2 (Fig. 2). It is also important to emphasize that the "extended" Fibonacci numbers F_n with the *even* indexes (n = 0, ±2, ±4, ±6, …) are "inscribed" into the graph of the hyperbolic Fibonacci sine $sF(x)$ in the discrete points (x = 0, ±2, ±4, ±6, …) and the "extended" Fibonacci numbers with the *odd* indexes (n = ±1, ±3, ±5, …) are "inscribed" into the hyperbolic Fibonacci cosine $cF(x)$ in the discrete points (x = ±1, ±3, ±5 …). On the other hand, the "extended" Lucas numbers with the *even* indexes are "inscribed" into the graph of the hyperbolic Lucas cosine $cL(x)$ in the discrete points (x = 0, ±2, ±4, ±6 …), and the "extended" Lucas numbers with the *odd* indexes are "inscribed" into the graph of the hyperbolic Lucas cosine $sL(x)$ in the discrete points (x = ±1, ±3, ±5 …).

These arguments lead us to the conclusion that the property of *recursiveness* is the main distinctive feature of the hyperbolic Fibonacci and Lucas functions (11) - (14), compared to the classical hyperbolic functions (7), (8). Thus, the hyperbolic Fibonacci and Lucas functions (11) - (14) are a new class of hyperbolic functions described in [17]. And we have a right to name these functions as *recursive hyperbolic functions*.

3.6 The Hyperbolic and Recursive Properties of the Hyperbolic Fibonacci and Lucas Functions

Thus, the hyperbolic functions (11) - (14) have distinctive mathematical properties compared to the classical hyperbolic functions (7), (8). First of all, they retain all well-known properties of the classical hyperbolic functions (7), (8) (*hyperbolic properties*), secondly, they have new unusual properties inherent to the Fibonacci and Lucas numbers (*recursive properties*).

We begin from the *hyperbolic properties*. First of them is *parity property*:

$$\underline{\textbf{Parity property}}:\quad \begin{cases} sF(-x) = -sF(x); \quad cF(-x) = cF(x) \\ sL(-x) = -sL(x); \quad cL(-x) = cL(x) \end{cases} \tag{21}$$

The relationship $\left[ch(x)\right]^2 - \left[sh(x)\right]^2 = 1$ is possibly one of the most important properties of the classical hyperbolic functions (7), (8). For the recursive hyperbolic functions (11) - (14), this property is given by Theorem 1 [17].

Theorem 1. The following relationships, similar to the relationship $\left[ch(x)\right]^2 - \left[sh(x)\right]^2 = 1$, are valid for the recursive hyperbolic Fibonacci and Lucas functions:

$$\left[cF(x)\right]^2 - \left[sF(x)\right]^2 = \frac{4}{5}. \tag{22}$$

$$\left[cL(x)\right]^2 - \left[sL(x)\right]^2 = 4. \tag{23}$$

Let us consider the examples of the *recursive properties* of the functions (11) - (14) [17].

Theorem 2. The following relations, which are similar to the recursive relations for the Fibonacci and Lucas numbers $F_{n+2} = F_{n+1} + F_n$ and $L_{n+2} = L_{n+1} + L_n$, are valid for the recursive hyperbolic Fibonacci and Lucas functions:

$$\begin{array}{l} \underline{\textbf{Recurrence relation for the Fibonacci hyperbolic functions}}: \\ sF(x+2) = cF(x+1) + sF(x) \\ cF(x+2) = sF(x+1) + cF(x) \\ \underline{\textbf{Recurrence relation for the Lucas hyperbolic functions}}: \\ sL(x+2) = cL(x+1) + sL(x) \\ cL(x+2) = sL(x+1) + cL(x) \end{array} \tag{24}$$

Theorem 3 (a generalization of Cassini's formula for continues domain). The following relations, which are similar to Cassini's formula $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1}$, are valid for the recursive hyperbolic Fibonacci functions:

$$\underline{\text{Cassini's formula}}: \quad \begin{aligned} &\left[sF(x)\right]^2 - cF(x+1)\,cF(x-1) = -1 \\ &\left[cF(x)\right]^2 - sF(x+1)\,sF(x-1) = 1 \end{aligned} \tag{25}$$

3.7 Theory of Fibonacci Numbers as a "Degenerate" Case of the Theory of the Recursive Hyperbolic Fibonacci and Lucas Functions

As follows from (15), (16), the two *"continuous"* identities for the recursive hyperbolic Fibonacci and Lucas functions (11) – (14) always correspond to one *"discrete"* identity for the "extended" Fibonacci and Lucas numbers (see Table 1). Conversely, we can obtain the *"discrete"* identity for the "extended" Fibonacci and Lucas numbers by using two corresponding *"continuous"* identities for the recursive hyperbolic Fibonacci and Lucas functions (11) – (14). As the "extended" Fibonacci and Lucas numbers, according to (15) and (16), are the *"discrete"* cases of the recursive hyperbolic Fibonacci and Lucas functions (11) – (14), this means that due the introduction of the *recursive hyperbolic Fibonacci and Lucas functions* (11) - (14) [17], the classical *"theory of Fibonacci numbers"* [18-20] as if *"degenerates,"* because this theory is a partial *("discrete"*) case of the more general (*"continuous*) theory of the *recursive hyperbolic Fibonacci and Lucas functions* (11) - (14). This conclusion is another unexpected result, which follows from the theory of the recursive hyperbolic Fibonacci and Lucas functions [17]. Such approach requires a **revision** of the existed "theory of Fibonacci numbers" [18-20] from the point of view of the theory of the *recursive hyperbolic Fibonacci and Lucas functions* (11) - (14).

However, a new geometric theory of phyllotaxis, created by the Ukrainian researcher Oleg Bodnar [21,22], is the most brilliant confirmation of the *uniqueness* and *fundamental nature* of the *recursive hyperbolic Fibonacci and Lucas functions* (11) - (14).

4 Phyllotaxis Phenomenon and Bodnar's Geometry

4.1 Mystery of Phyllotaxis

As outlined in the chapter "Authority of Nature" of the book [23], the most important criterion for the evaluation of new mathematical results is *"its value to the sciences."* What is the significance of the *recursive hyperbolic Fibonacci and Lucas functions* (11) - (14) for modern science? A new geometric theory of phyllotaxis, created by Ukrainian researcher Oleg Bodnar [21,22], gives the answer to this question.

Among Nature's phenomena, which surround us, perhaps, the *botanical phenomenon of phyllotaxis* [21] is the best known and most common.

This phenomenon is inherent to many biological objects. The essence of phyllotaxis phenomenon consists in a spiral disposition of leaves on plant's stems of trees, petals in flower baskets, seeds in pine cone and sunflower discs etc (Fig. 3). This phenomenon, known already since **Kepler's** time, was a subject of discussion of many scientists and thinkers, including **Leonardo da Vinci, Turing, Weil** and others. In phyllotaxis phenomenon more complex concepts of symmetry, in particular, the concept of *helical symmetry*, are used.

On the surfaces of *phyllotaxis' objects*, their bio-organs (seeds on the sunflower's disks and pine cones etc.) are disposed in the form of the left-twisted and right-twisted spirals. For the evaluation of the symmetrical properties of such phyllotaxis' objects, it is used usually the number ratios for the left-twisted and right-twisted spirals, observed on the surfaces of the phyllotaxis' objects. Botanists proved that these ratios are equal to the ratios of the adjacent Fibonacci numbers, i.e.,

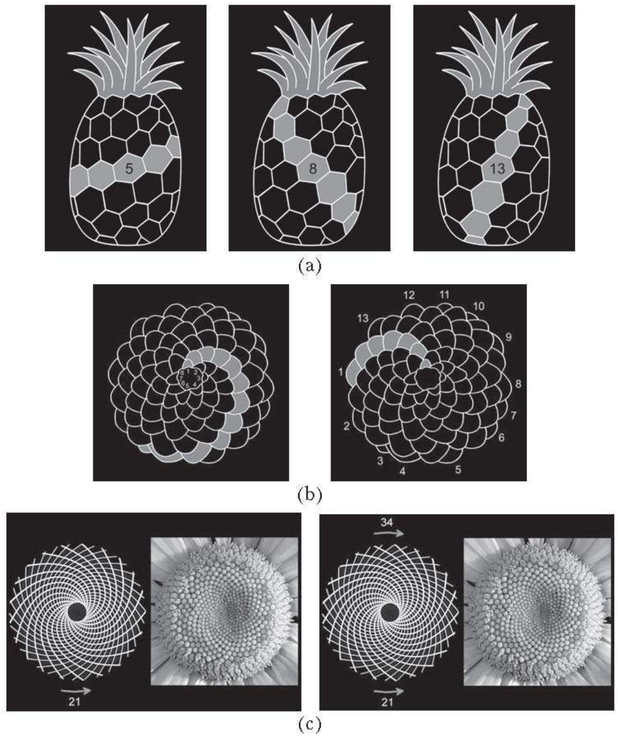

Fig. 3. Geometric models of phyllotaxis structures: (a) Pineapple; (b) Pine cone; (c) Head of sunflower

$$\frac{F_{n+1}}{F_n}: \quad \frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13},\ldots \to \Phi = \frac{1+\sqrt{5}}{2}. \tag{26}$$

The ratios (26) are called *phyllotaxis orders* [21,22]. They are various for different phyllotaxis' objects. For example, the disc of sunflower can have the phyllotaxis orders, given by the Fibonacci ratios $\frac{89}{55},\frac{144}{89}$ and even $\frac{233}{144}$.

By observing phyllotaxis structures in the completed form and by enjoying the well organized pictures on their surfaces (Fig. 3), we always ask the question: how are the Fibonacci spirals are formed on their surfaces during their growth? It is proved [21] that during the growth of the phyllotaxis' object, a natural modification (increasing) of symmetry orders happens and this modification of symmetry obeys to the law:

$$\frac{2}{1} \to \frac{3}{2} \to \frac{5}{3} \to \frac{8}{5} \to \frac{13}{8} \to \frac{21}{13} \to \ldots \tag{27}$$

The modification of the phyllotaxis orders according to (27) is named *dynamic symmetry* [21,22]. This problem, which has attracted the attention of **Kepler**, **Leonardo da Vinci**, **Turing** and **Weil**, rightfully can be called MILLENIUM PROBLEM.

4.2 Key Ideas of Bodnar's Geometry

Without going into the detailed description of Bodnar's geometry and referring to Bodnar's works [21,22], we will analyze only the key ideas of this geometry. According to (27), in the process of their growth, the phyllotaxis' objects pass through series of intermediate states, each of which corresponds to the certain order of symmetry (see (27)). Bodnar's geometry is based on the following assumptions:

1. The geometry of phyllotaxis is *hyperbolic geometry*.
2. A passage of phyllotaxis' object from any state to another one is realized by means of *hyperbolic rotation,* which is the main transformation of hyperbolic geometry.
3. The mathematical relations for phyllotaxis' objects are described by the *recursive hyperbolic Fibonacci functions* (11), (12). This assumption is most unusual, but just this assumption led Bodnar to very simple explanation of the *phyllotaxis mystery* (27).

A number of the important conclusions are following from Bodnar's geometry:

1. "Bodnar's geometry" opened for modern science a new "hyperbolic world," the *world of phyllotaxis*. The main feature of this world is the fact that the basic geometric relations of this world are described by the *recursive hyperbolic Fibonacci functions* (11), (12) what cause the appearance of the Fibonacci spirals on the surface of phyllotaxis' objects.
2. "Bodnar's geometry" showed that hyperbolic geometry is much more spread in the real world than it seemed before. **The recursive hyperbolic Fibonacci and Lucas functions (11), (12) are not the "fiction" of mathematicians; they are the "*natural*" *functions* of Nature.** They appear in different botanical structures such, as pine cones, pineapples, cacti, discs of sunflower and so on.
3. Bodnar's geometry is a new hyperbolic geometry of wildlife and this fact is of fundamental importance for the future development of such sciences as biology, botany, physiology, medicine, genetics, and so on.

5 Fibonacci λ-Numbers, Metallic Means, Gazale's Formulas and General Theory of Recursive Hyperbolic Functions

5.1 Fibonacci λ-Numbers

5.1.1 A brief of history

In the late 20 th and early 21st centuries, several researchers from different countries –Argentinean mathematician **Vera W. de Spinadel** [24], French mathematician **Midhat Gazale** [25], American mathematician **Jay Kappraff** [26], Russian engineer **Alexander Tatarenko** [27], Armenian philosopher and physicist **Hrant Arakelyan** [28], Russian researcher **Victor Shenyagin** [29], Ukrainian physicist **Nikolai Kosinov** [30], Spanish mathematicians **Sergio Falcon** and **Angel Plaza** [31] and others independently one to another began to study a new classes of the recurrence numerical sequences, which are a generalization of the classical Fibonacci numbers. These numerical sequences led to the discovery of a new class of mathematical constants, called *"metallic means"* by **Vera W. de Spinadel** [24].

The interest of many independent researchers from different countries (US, Canada, Argentina, France, Spain, Russia, Armenia, Ukraine) can not be accidental. This means that the problem of the generalization of Fibonacci numbers and "golden ratio" has matured in modern science.

5.1.2 The recurrence relation for the Fibonacci λ-numbers

Let us give an integer $\lambda = 1, 2, 3, \ldots$ and consider the following recurrence relation:

$$F_\lambda(n+2) = \lambda F_\lambda(n+1) + F_\lambda(n);\ F_\lambda(0) = 0,\ F_\lambda(1) = 1. \tag{28}$$

The recurrence relation (28) generates an infinite number of new numerical sequences, because every integer $\lambda = 1, 2, 3, \ldots$ generates its own recursive numerical sequence.

Basing on the fact, that for the case λ=1 the recurrence relation (28) generates the classical Fibonacci numbers, we will name a general class of the numerical sequences, generated by the recurrence relation (28), the *Fibonacci λ-numbers*.

Note that for the case λ=2 the recurrence relation (28) generates the so-called *Pell numbers* [32]:

$$0, 1, 2, 5, 12, 29, 70, 169, 408, \ldots \tag{29}$$

5.1.3 The generalized Cassini's formula for the "extended" Fibonacci λ-numbers

Table 2 shows the examples of the "extended" Fibonacci λ-numbers

Table 2. The "extended" Fibonacci λ-numbers for the cases λ=1,2,3,4

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$F_1(n)$	0	1	1	2	3	5	8	13	21
$F_1(-n)$	0	1	-1	2	-3	5	-8	13	-21
$F_2(n)$	0	1	2	5	12	29	70	169	408
$F_2(-n)$	0	1	-2	5	-12	29	-70	169	-408
$F_3(n)$	0	1	3	10	33	109	360	1189	3927
$F_3(-n)$	0	1	-3	10	-33	109	-360	1189	-3927
$F_4(n)$	0	1	4	17	72	305	1292	5473	23184
$F_4(-n)$	0	1	-4	17	-72	305	-1292	5473	-23184

In the article [33], the surprising mathematical formula, which is a generalization of Cassini's formula (4) for the classical Fibonacci numbers, has been proved:

$$\underline{\textbf{Generalized Cassini's formula}}: \quad F_\lambda^2(n) - F_\lambda(n-1)F_\lambda(n+1) = (-1)^{n+1}, \tag{30}$$

where $\lambda = 1, 2, 3, \ldots$ is a given integer, $F_\lambda(n-1), F_\lambda(n), F_\lambda(n+1)$ are the adjacent Fibonacci λ-numbers.

The formula (30) sounds as follows:

> *"The quadrate of any Fibonacci λ-number $F_\lambda(n)$ are always different from the product of the two adjacent Fibonacci λ-numbers $F_\lambda(n-1)$ and $F_\lambda(n+1)$, which surround the initial Fibonacci λ-number $F_\lambda(n)$, by the number 1; herewith the sign of the difference of 1 depends on the parity of n: if n is even, then the difference of 1 is taken with the sign "minus," otherwise, with the sign "plus".*

As is known, a study of integer sequences is the area of number theory. The Fibonacci λ-numbers, given by the recurrence relation (28), are integer sequences. Therefore, for many mathematicians in the field of number theory, the existence of the infinite number of the integer sequences, which satisfy to the surprising generalized Cassini's formula (30) [33], may be a big surprise.

5.2 The "Metallic Means" by Vera W. de Spinadel

5.2.1 Definition

The following characteristic equation follows from the recurrence relation (28):

$$x^2 - \lambda x - 1 = 0. \tag{31}$$

The algebraic equation (31) has the following roots:

$$x_1 = \frac{\lambda + \sqrt{4+\lambda^2}}{2} \tag{32}$$

$$x_2 = \frac{\lambda - \sqrt{4+\lambda^2}}{2} \tag{33}$$

Denote the positive root (32) through Φ_λ, i.e.,

$$\Phi_\lambda = \frac{\lambda + \sqrt{4+\lambda^2}}{2} \tag{34}$$

Note what for the case $\lambda = 1$ the formula (34) is reduced to the formula for the *golden ratio*:

$$\Phi_1 = \frac{1+\sqrt{5}}{2}. \tag{35}$$

This means that the formula (35) gives a wide class of the new mathematical constants, which are a generalization of the golden ratio (35).

Basing on this analogy, the Argentinean mathematician **Vera W. de Spinadel** named in [24] the mathematical constants (34) the *metallic means*. If we take λ =1, 2, 3, 4 in (34), then we get the following mathematical constants having according to **Vera de Spinadel** the following special names:

$$\Phi_1 = \frac{1+\sqrt{5}}{2} \left(\text{the Golden Mean}, \lambda = 1\right);$$
$$\Phi_2 = 1+\sqrt{2} \left(\text{the Silver Mean}, \lambda = 2\right);$$
$$\Phi_3 = \frac{3+\sqrt{13}}{2} \left(\text{the Bronze Mean}, \lambda = 3\right);$$
$$\Phi_4 = 2+\sqrt{5} \left(\text{the Cooper Mean}, \lambda = 4\right).$$

Other metallic means ($\lambda \geq 5$) do not have special names:

$$\Phi_5 = \frac{5+\sqrt{29}}{2}; \quad \Phi_6 = 3+2\sqrt{10}; \quad \Phi_7 = \frac{7+2\sqrt{14}}{2}; \quad \Phi_8 = 4+\sqrt{17}.$$

5.2.2 The simplest algebraic properties of the "metallic means."

It follows from the algebraic equation (31) the following simple algebraic properties of the *metallic means* (34):

$$\lambda = \Phi_\lambda - \frac{1}{\Phi_\lambda}, \tag{36}$$

$$\Phi_\lambda + \frac{1}{\Phi_\lambda} = \sqrt{4+\lambda^2} \tag{37}$$

$$\Phi_\lambda^n = \lambda \Phi_\lambda^{n-1} + \Phi_\lambda^{n-2}, \tag{38}$$

where n=0, ±1, ±2, ±3, … ,

$$\Phi_\lambda = \sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\lambda\sqrt{1+\ldots}}}} \tag{39}$$

$$\Phi_\lambda = \lambda + \cfrac{1}{\lambda + \cfrac{1}{\lambda + \cfrac{1}{\lambda + \ldots}}} \tag{40}$$

Note that for the case $\lambda = 1$ the representations (39) and (40) coincide with the well known representations of the classical golden ratio in the forms:

$$\Phi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}; \quad \Phi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\ldots}}}. \tag{41}$$

The representations of the "metallic means" in the forms (39) and (40), similar to the surprising representations (41), are additional confirmations of the fact that the "metallic means" Φ_λ are new mathematical constants!

5.3 Gazale's Formulas

5.3.1 Gazale's formula for the Fibonacci λ-numbers.

The formula (28) defines the Fibonacci λ-numbers $F_\lambda(n)$ recursively. However, **Midhat Gazale** in the book [25] represents the "extended" Fibonacci λ-numbers $F_\lambda(n)$ in the explicit form through the "metallic mean" Φ_λ:

$$F_\lambda(n) = \frac{\Phi_\lambda^n - (-1/\Phi_\lambda)^n}{\sqrt{4+\lambda^2}} \tag{42}$$

Note that for the partial case $\lambda = 1$, the formula (42) is reduced to the Binet's formula for Fibonacci numbers.

5.3.2 Self-similarity and Gazale's hypothesis

A conception of *self-similarity* [34] is spread widely in Nature, sciences and mathematics. As pointed in the article [34], "*in* mathematics*, a **self-similar** object is exactly or approximately similar to a part of itself (i.e. the whole has the same shape as one or more of the parts). Many objects in the real world, such as coastlines, are statistically self-similar: parts of them show the same statistical properties at many scales. Self-similarity is a typical property of fractals. Scale invariance is an exact form of self-similarity where at any magnification there is a smaller piece of the object that is similar to the whole*".

All phyllotaxis' objects [21] are brilliant examples of *self-similarity*. In particular, the ratios of Fibonacci numbers in the sequence (27) are examples of *self-similarity.* This means that a growth of phyllotaxis objects, according to the regularity (27), is based on the *self-similarity principle*. Also Bodnar's geometry [21,22], which explains the growth of phyllotaxis objects, is based on *self-similarity principle.*

In mathematics, *self-similarity* is expressed through *recursive relations.*

The central notion of Gazale's book [25] is the notion of *self-similarity*. Gazale was one of the first who begun to study Fibonacci λ-numbers. The derivation of mathematical formula (42), which expresses Fibonacci λ-numbers through the "metallic means," is one of the main Gazale's mathematical achievements, described in the book [25]. In the book [25], Gazale put forward the following unusual hypothesis, which has direct relation to mathematical models of self-similarity:

> **Gazale's hypothesis:** "*The numerical sequence $F_{m,n+2} = F_{m,n} + mF_{m,n+1}$, which I call here the Fibonacci sequence of the order m, play a key role in the study of self-similarity*".

If we take in this formula that $m = \lambda$, $F_{m,n+2} = F_\lambda(n+2)$, $F_{m,n} = F_\lambda(n)$ $F_{m,n+1} = F_\lambda(n+1)$, then we get the recurrence relation (28) for the Fibonacci λ-numbers.

This means that the recurrence relation (28), which gives the Fibonacci λ-numbers, according to *Gazale's hypothesis*, expresses the *self-similarity principle,* which is one of the most important principles of Nature, sciences and mathematics.

5.4 Hyperbolic Fibonacci and Lucas λ-Functions

5.4.1 Definition

The researches by **Vera de Spinadel** [24], **Midhat Gazale** [25], **Jay Kappraff** [26] and others have became for **Alexey Stakhov** a launching pad for the creation of the general theory of recursive hyperbolic functions, described in the works [35,36].

In order to determine a new class of hyperbolic functions, **Alexey Stakhov** represent in [35,36] *Gazale's formulas* for the Fibonacci and Lucas λ-numbers in the following form:

$$F_\lambda(n)=\begin{cases}\dfrac{\Phi_\lambda^n-\Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}} & \text{for} \quad n=2k\\ \dfrac{\Phi_\lambda^n+\Phi_\lambda^{-n}}{\sqrt{4+\lambda^2}} & \text{for} \quad n=2k+1\end{cases} \tag{43}$$

$$L_\lambda(n)=\begin{cases}\Phi_\lambda^n-\Phi_\lambda^{-n} & \text{for} \quad n=2k+1\\ \Phi_\lambda^n+\Phi_\lambda^{-n} & \text{for} \quad n=2k\end{cases} \tag{44}$$

Comparing *Gazale's formulas* (43) and (44) to the classical hyperbolic functions (7), (8), we can see their similarity by mathematical structures. This similarity became a reason to introduce a general class of hyperbolic functions called in [35,36] the *hyperbolic Fibonacci and Lucas* λ*-functions:*

Hyperbolic Fibonacci λ-sine and λ-cosine

$$sF_\lambda(x)=\frac{\Phi_\lambda^x-\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}==\frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x-\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{45}$$

$$cF_\lambda(x)=\frac{\Phi_\lambda^x+\Phi_\lambda^{-x}}{\sqrt{4+\lambda^2}}==\frac{1}{\sqrt{4+\lambda^2}}\left[\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x+\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\right] \tag{46}$$

Hyperbolic Lucas λ-sine and λ-cosine

$$sL_\lambda(x)=\Phi_\lambda^x-\Phi_\lambda^{-x}=\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x-\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x} \tag{47}$$

$$cL_\lambda(x)=\Phi_\lambda^x+\Phi_\lambda^{-x}=\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^x+\left(\frac{\lambda+\sqrt{4+\lambda^2}}{2}\right)^{-x}\quad, \tag{48}$$

where x is continuous variable and $\lambda=1,2,3,...$ is a given integer.

It is easy to see that the functions (45), (46) and (47), (48) are connected by very simple relations:

$$sF_\lambda(x)\quad=\quad\frac{sL_\lambda(x)}{\sqrt{4+\lambda^2}};\quad cF_\lambda(x)\quad=\quad\frac{cL_\lambda(x)}{\sqrt{4+\lambda^2}}. \tag{49}$$

5.4.2 An uniqueness of the hyperbolic Fibonacci and Lucas λ-functions

It should be noted the following unique properties of the hyperbolic Fibonacci and Lucas λ- functions (45)–(48):

1. The hyperbolic Fibonacci and Lucas λ-functions (45)–(48) are, on the one hand, a generalization of the classical hyperbolic functions (7), (8), but on the other hand, a generalization of the recursive hyperbolic Fibonacci and Lucas functions (11)–(14), which are a partial case of the functions (45)–(48) for the case $\lambda=1$.

2. Their uniqueness consists of the fact that they, on the one hand, retain all *hyperbolic properties*, inherent for the classical hyperbolic functions (7), (8). On the other hand, they have *recursive properties*, inherent to the recursive hyperbolic Fibonacci and Lucas functions (11)–(14).
3. The next unique feature of the functions (45)–(48) is the fact that the general formulas (45)–(48) define theoretically infinite number of new classes of the recursive hyperbolic functions, because every integer $\lambda = 1,2,3,...$ generates a new, previously unknown class of the recursive hyperbolic functions.
4. One more unique feature of the functions (45)–(48) is their deep connection to the "extended" Fibonacci and Lucas λ-numbers, defined by *Gazale's formulas* (43), (44). This connection is determined identically by the following relations:

$$\begin{cases} F_\lambda(n) = \begin{cases} sF_\lambda(n), & n = 2k \\ cF_\lambda(n), & n = 2k+1 \end{cases} \\ L_\lambda(n) = \begin{cases} cL_\lambda(n), & n = 2k \\ sL_\lambda(n), & n = 2k+1 \end{cases} \end{cases}. \tag{50}$$

5. According to *Gazale's hypothesis*, the recursive hyperbolic functions (45)–(48), following from *Gazale's formulas* (43), (44), express the *similarity principle*, which is the most important principle of Nature, science and mathematics.

5.4.3 Hyperbolic and recursive properties of the hyperbolic Fibonacci and Lucas λ-functions

As examples of hyperbolic properties of the functions (45)–(48), we consider the **parity properties** and the **analog of the Pythagoras Theorem**:

Parity properties

$$\begin{aligned} &sF_\lambda(-x) = -sF_\lambda(x);\, cF_\lambda(-x) = cF_\lambda(x) \\ &sL_\lambda(-x) = -sL_\lambda(x);\; cL_\lambda(-x) = cL_\lambda(x) \end{aligned}. \tag{51}$$

Analog of the Pythagoras Theorem

$$\begin{aligned} &\left[cF_\lambda(x)\right]^2 - \left[sF_\lambda(x)\right]^2 = \frac{4}{4+\lambda^2} \\ &\left[cL_\lambda(x)\right]^2 - \left[sL_\lambda(x)\right]^2 = 4 \end{aligned} \tag{52}$$

Some *recursive properties* of functions (45)–(48) are given by the following theorems, proved in [35,36].

Theorem 1. The following relations, which are similar to the recurrence relation for the Fibonacci λ-numbers $F_\lambda(n+2) = \lambda F_\lambda(n+1) + F_\lambda(n)$, are valid for the hyperbolic Fibonacci λ-functions:

$$\begin{aligned} sF_\lambda(x+2) &= \lambda cF_\lambda(x+1) + sF_\lambda(x), \\ cF_\lambda(x+2) &= \lambda sF_\lambda(x+1) + cF_\lambda(x). \end{aligned} \tag{53}$$

Theorem 2. (the generalized Cassini's formula for continuous domain). The following relations, which are similar to the generalized Cassini's formula for the Fibonacci λ-numbers $F_\lambda^2(n) - F_\lambda(n-1)F_\lambda(n+1) = (-1)^{n+1}$, are valid for the hyperbolic Fibonacci λ-functions:

$$\begin{aligned} \left[sF_{\lambda}(x)\right]^2 - cF_{\lambda}(x+1)\,cF_{\lambda}(x-1) &= -1, \\ \left[cF_{\lambda}(x)\right]^2 - sF_{\lambda}(x+1)\,sF_{\lambda}(x-1) &= 1. \end{aligned} \tag{54}$$

6 Original Solution to Hilbert's Fourth Problem

6.1 General Considerations

The basis of the original solution to Hilbert's Fourth Problem, resulting in the works [6-10], is the approach, named in [14] *"the game of the functions."* The essence of this approach consists in the fact that we are remaining in the framework of the classical hyperbolic geometry, that is, we do not change its postulates; however, for the description of the mathematical relationships of the new hyperbolic geometry, we use new classes of recursive hyperbolic functions, introduced in the works [17,35,36]. Note that the Ukrainian researcher Oleg Bodnar was the first scientist, who used this approach to create a geometric theory of phyllotaxis [21,22]. Until the publication of the articles [17,35,36], such approach to the creation of new hyperbolic geometries cannot be used, because the new classes of hyperbolic functions, having *recursive properties*, were unknown.

In this connection, the creation of Bodnar's geometry [21,22], which relates to a new kind of hyperbolic geometry, became a brilliant confirmation of fruitfulness of the new approach to the solution of Hilbert Fourth Problem, because the replacement of the classical hyperbolic functions (7), (8) on the recursive hyperbolic Fibonacci functions (11)–(12) underlies Bodnar's geometry.

6.2 An Original Solution of Hilbert's Fourth Problem and "Golden" Hyperbolic Geometry

6.2.1 A general idea

In the articles [35,36], the wide generalization of the recursive hyperbolic Fibonacci and Lucas functions (11)–(14) is presented. Here the recursive hyperbolic Fibonacci and Lucas λ-functions (45)–(48), which extend the class of the recursive hyperbolic functions ad infinitum, are described. These new classes of the surprising recursive hyperbolic functions, based on Spinadel's *metallic means* [24] and *Gazale's formulas* [25], became the basis of the original solution to Hilbert's Fourth Problem [14–19].

The following **general idea** underlies the original solution to Hilbert's Fourth Problem [14–19]:

Every class of the recursive hyperbolic functions (45)–(48) generates new hyperbolic geometry.

It follows from this statement that **the number of new hyperbolic geometries, following from such approach, is theoretically infinite**. We will name these new *recursive hyperbolic geometries*, based on the *self-similarity principle*, with the common title of the "**Golden" Hyperbolic Geometry**.

Thus, the "Golden" Hyperbolic Geometry has two distinctive features:

1. This geometry is **fractal geometry**, based on the recursive Fibonacci λ-numbers (28).
2. The **principle of self-similarity** underlies this new hyperbolic geometry.

6.2.2 The metric λ-forms of Lobachevski's plane

There exists in hyperbolic geometry a notion of the *metric form of Lobachevski's plane,* which is based on the classical hyperbolic functions. Developing this concept, the following formula for the *metric form of Lobachevski's plane*, based on the hyperbolic Fibonacci λ-functions (45), (46), has been derived in [6–10]:

$$(ds)^2 = \ln^2(\Phi_\lambda)(du)^2 + \frac{4+\lambda^2}{4}\left[sF_\lambda(u)\right]^2(dv)^2, \tag{55}$$

where $\Phi_\lambda = \frac{\lambda+\sqrt{4+\lambda^2}}{2}$ is the *metallic means* and $sF_\lambda(u)$ is the recursive hyperbolic Fibonacci λ-sine (45), λ=1,2,3,…. The forms (55) are called the *metric* λ*-forms of Lobachevski's plane* [6–10].

The formula (55) gives an infinite number of new *Lobachevski's geometries* (*"golden," "silver," "bronze," "cooper"* and so on ad infinitum) according to the used class of the recursive hyperbolic Fibonacci λ-functions (45), (46).

The formula (55) gives an infinite number of the *metric forms of Lobachevski's plane*. This means that there is infinite number of the new hyperbolic geometries, which are based on the *metallic means* (34). These new hyperbolic geometries *"with equal right, stand next to Euclidean geometry"* (**David Hilbert**). Thus, the formula (55) can be considered as the original solution to Hilbert's Fourth Problem, based on the *"game of functions"* [14]. There are an infinite number of the new hyperbolic geometries, described by the formula (55), which are close to Euclidean geometry. Every of these geometries manifests itself in Fibonacci λ-numbers (28), which can appear in physical world similarly to *Bodnar's hyperbolic geometry* [21,22], where the classical Fibonacci numbers appear at the surface of phyllotaxis' objects.

6.3 A New Challenge to Theoretical Natural Sciences

Thus, the main result of the research, described in [6-10], is a proof of the existence of an infinite number of the recursive hyperbolic λ-functions (45) - (48), based on the *metallic means* (34). In addition, for the given λ=1,2,3,.. every class of the recursive hyperbolic λ-functions, given by (45) - (48), generates its own recursive hyperbolic geometry, which leads to the appearance of the "physical worlds" with specific properties, which are determined by the *metallic means* (34). The new geometric theory of phyllotaxis, created by Oleg Bodnar [21,22], is the striking example of this. Bodnar proved that the "world of phyllotaxis" is a specific "hyperbolic world," in which the *hyperbolicity* manifests itself in the *Fibonacci spirals* on the surface of phyllotaxis' objects.

However, the "golden" hyperbolic Fibonacci functions (11), (12), which underlie the *"hyperbolic world of phyllotaxis"* [21], are a special case of the recursive hyperbolic Fibonacci λ-functions (45), (46) for λ=1. In this regard, we have all reasons to suppose that other types of the recursive hyperbolic λ-functions (45), (46), based on the *metallic means,* can be good models for the new "hyperbolic worlds" that can really exist in Nature. Modern science cannot find these special "hyperbolic worlds," because the recursive hyperbolic functions (45), (46) were unknown until now [16,17,35,36]. Basing on the success of *Bodnar's hyperbolic geometry* [21,22], we can put forward in front to theoretical physics, chemistry, crystallography, botany, biology, genetics and other branches of theoretical natural sciences the **challenge for searching of the *new hyperbolic worlds of Nature,* based on the new classes of the recursive hyperbolic λ-functions (45), (46).**

Studying the recursive hyperbolic functions (45), (46), we can assume that the recursive hyperbolic λ-functions with the bases

$$\Phi_1 = \frac{1+\sqrt{5}}{2} \left(\text{the Golden Mean}, \lambda = 1\right);$$
$$\Phi_2 = 1+\sqrt{2} \left(\text{the Silver Mean}, \lambda = 2\right);$$
$$\Phi_3 = \frac{3+\sqrt{13}}{2} \left(\text{the Bronze Mean}, \lambda = 3\right);$$
$$\Phi_4 = 2+\sqrt{5} \left(\text{the Cooper Mean}, \lambda = 4\right).$$

are of the of greatest interest for theoretical natural sciences.

A class of the "*golden" recursive hyperbolic Fibonacci functions*

$$sF(x) = \frac{\Phi^x - \Phi^{-x}}{\sqrt{5}}; \quad cF(x) = \frac{\Phi^x + \Phi^{-x}}{\sqrt{5}}, \tag{56}$$

based on the classical *golden ratio*, plays the leading role among them. These functions underlie *Bodnar's geometry* [21,22], a *new hyperbolic geometry of phyllotaxis*.

The next candidate for the new "hyperbolic world" of Nature (after "Bodnar's hyperbolic geometry" [21,22]) may be, for example, *silver hyperbolic functions*:

$$\begin{aligned} sF_2(x) &= \frac{\Phi_2^x - \Phi_2^{-x}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x - \left(1+\sqrt{2}\right)^{-x}\right]; \\ cF_2(x) &= \frac{\Phi_2^x + \Phi_2^{-x}}{\sqrt{8}} = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^x + \left(1+\sqrt{2}\right)^{-x}\right], \end{aligned} \tag{57}$$

which are connected with *Pell numbers* [32] and are based on the *silver proportion* $\Phi_2 = 1+\sqrt{2}$, connected with the fundamental mathematical constant $\sqrt{2}$.

7 Conclusion: Whether are New Classes of Hyperbolic Functions and the Following from Them Original Solution to Hilbert's Fourth Problem Fundamental Scientific Result?

In his classic work *The Analytical Theory of Heat* (1822), the prominent French mathematician and physicist **Jean-Baptiste Joseph Fourier** (1768 - 1830) expressed his opinion about the application of mathematics to physical problems (the quote is taken from [23]):

> *"The profound study of nature is the most fertile source of mathematical discoveries. This study offers not only the advantages of a well determined goal but the advantage of excluding vague questions and useless calculations. It is a means of building analysis itself and of discovering the ideas which matter most and which science must always preserve.* ***The fundamental ideas are those which represent the natural happenings…***
>
> *Its chief attribute is clarity; it has no symbols to express confused ideas. It brings together the most diverse phenomena and discovers hidden analogies which unite them".*

The above quote of the eminent mathematician and physicist has direct relevance to the researches, described in this article.

We begin from the articles [16,17]. These articles contain a description of very important mathematical discovery, the *recursive hyperbolic functions*, based on the *golden ratio, Fibonacci and Lucas numbers*.

In Bodnar's works [21,22], the similar functions have been used for the creation of new geometric theory of phyllotaxis what is a botanical phenomenon, well-known since Johannes Kepler's time.

Bodnar's geometry is a new kind of Lobachevski's geometry. This geometry uncovers a mystery of phyllotaxis, that is, describes how the *dynamic symmetry* of phyllotaxis' object is changing in the process of its growth. This is expressed in changing the structure of Fibonacci's spirals on the surface of phyllotaxis' object.

Thus, *Bodnar's geometry* showed that the world of phyllotaxis is the hyperbolic world, based on hyperbolic Fibonacci and Lucas functions. That is, these functions are not "fiction" of mathematicians; they are natural functions that are used in natural objects during millions, and perhaps billions of years.

In Stakhov's articles [35,36], the new classes of the recursive hyperbolic functions based on *Spinadel's metallic means* [24] and *Gazale's formulas* [25] have been introduced. The new classes of the recursive hyperbolic functions are a generalization of *"golden" recursive hyperbolic functions* (11) - (14). In Stakhov&Aranson's works [6-10], these functions have been used to produce the unexpected solution to Hilbert's Fourth Problem, which relates to the **"Golden" Hyperbolic Geometry**. This solution puts forward in front to theoretical natural sciences the fundamental challenge searching new "hyperbolic worlds of nature".

As follows from the above arguments; all of these studies relate directly to Nature and theoretical natural sciences. That is why, the new classes of the *recursive hyperbolic functions* [16,17,35,36] together with *Bodnar's geometry* [21,22] and the original solution to *Hilbert's Fourth Problem* [6-10], according to **Fourier**, rightfully can be attributed to the category of **fundamental scientific discoveries** and, as such, they will remain in modern and future sciences.

Taking into consideration the importance of Hilbert's Fourth Problem not only for geometry, but also for the entire theoretical natural sciences, the authors of this article take the courage to assert that Hilbert's Fourth Problem was not included into the list of the MILLENNIUM PROBLEMS, by mistake (the modern mathematicians could not understand and evaluate this problem) and therefore **Hilbert's Fourth Problem deserves to be recognized as the MILLENNIUM PROBLEM in geometry.**

Competing Interests

Authors have declared that no competing interests exist.

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