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Some Fixed Point Results of Rational Type-Contraction Mapping in S-Metric Space

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we demonstrate the existence of some fixed points of rational type contraction in context of S-metric space and we examine the T-stability of the P-property for some mapping. Also, we present few examples to illustrate the validity of the results obtained in the paper.

Keywords: Fixed point, rational type contraction, S-metric space.

MSC: 54H25, 47H10.

1 Introduction and Preliminaries

Fixed point theory is an active area of research with various application in real life. The Banach contraction principle [1] is a significant consequence in fixed-point theory. Several authors have refined this well-known principle which can be studied in (refer [2], [3], [4], [5], [6]). One of the main approaches used in this theory to demonstrate the existence and uniqueness of fixed point is contraction (see [7], [8], [9], [10], [11], [12]). In1989,

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Bakhtin [13] was the first who introduced the concept of b-metrics pace. In 1993, Czerwik [14] extended the results of Bakhtin [15] and gave generalization of Banach fixed point theorem in b-metric spaces. In 2012, the idea of S- metric space was established by Sedghi et al. [16], who also proved fixed point theorems there in. More well-known results in the direction of S-metric space are involved in (refer [17-20]).

Furthermore, we proceed by reviewing some important definitions and key terms that would be used throughout our discussion.

Definition 1.1 [16]: "Let $\mathcal{X}$ be a non-empty set. An S-metric on $\mathcal{X}$ is a mapping $\mathcal{S}: \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ which satisfies the following condition:

($\mathcal{S}_1$) $\mathcal{S}(u, v, w) = 0$ if and only if $u = v = w = 0$;

($\mathcal{S}_2$) $\mathcal{S}(u, v, w) \leq \mathcal{S}(u, u, a) + \mathcal{S}(v, v, a) + \mathcal{S}(w, w, a)$, for all $u, v, w, a \in \mathcal{X}.$

The pair $(\mathcal{X}, \mathcal{S})$ is called an S-metric space."

Example 1.2 [16]: "Let $\mathcal{X} = \mathbb{R}$. Then $\mathcal{S}(u, v, w)$ is an S-metric on $\mathbb{R}$ given by $\mathcal{S}(u, v, w) = |u - w| + |v - w|$, which is known as usual S-metric space on $\mathcal{X}$."

Manoj K. et al. [21], proved fixed point theorem by using altering distance function in S-metric space.
Theorem 1.3 [21]: "Let $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a mapping on a complete S-metric space $(\mathcal{X}, \mathcal{S})$ such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq \lambda\, \mathcal{S}(u, u, v) + \eta\, \frac{\mathcal{S}(u,u,\mathcal{T}v)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)},$$

for all $u, v \in X$, $\lambda, \eta > 0$, $\lambda + \eta < 1$. Then $\mathcal{T}$ possess a fixed point $w \in \mathcal{X}$ which is unique."

Theorem 1.4 [21]: "Let $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a mapping on a complete S-metric space $(\mathcal{X}, \mathcal{S})$ such that

$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq \lambda\, \mathcal{S}(u, u, v) + \eta\, \frac{\mathcal{S}(v,v,\mathcal{T}v)[1+\mathcal{S}(u,u,\mathcal{T}u)]}{1+\mathcal{S}(u,u,v)}$, for all $u, v \in X$, $\lambda, \eta > 0$, $\lambda + \eta < 1$. Then $\mathcal{T}$ possess a fixed point $w \in \mathcal{X}$ which is unique [22]."

Lemma 1.5 [16]: "If $(\mathcal{X}, \mathcal{S})$ is an S-metric space on a non-empty set $\mathcal{X}$, then $(\mathcal{X}, \mathcal{S})$ satisfy the symmetric condition, that is $\mathcal{S}(u, u, v) = \mathcal{S}(v, v, u)$, for all $u, v \in \mathcal{X}$."

Definition 1.6 [16] "Let $(\mathcal{X}, \mathcal{S})$ be an S-metric space. For $r > 0$ and $u \in \mathcal{X}$ we define the open ball $B_s(u, r)$ and closed ball and $B_s[u, r]$ with a center u and radius r as follows:

$B_s(u, r) = \{v \in \mathcal{X}: \mathcal{S}(v, v, u) < r\}$

$B_s[u, r] = \{v \in \mathcal{X}: \mathcal{S}(v, v, u) \leq r\}$."

Definition 1.7 [17]: "A sequence $\{u_n\}$ in $(\mathcal{X}, \mathcal{S})$ is said to be convergent to some point $u \in \mathcal{X}$, if $\mathcal{S}(u_n, u_n, u) \to 0$ as $n \to \infty$."

Definition 1.8 [17]: "A sequence $\{u_n\}$ in $(\mathcal{X}, \mathcal{S})$ is said to be Cauchy sequence if $\mathcal{S}(u_n, u_n, u_m) \to 0$ as $n, m \to \infty$."

Definition 1.9 [17]: "An S -metric space $(\mathcal{X}, \mathcal{S})$ is said to be complete if every Cauchy sequence in X is convergent in $\mathcal{X}$."

Lemma 1.10 [17]: "Let $(\mathcal{X}, \mathcal{S})$ be an S-metric space. If $u_n \to u$ *and* $v_n \to v$ then $\mathcal{S}(u_n, u_n, v_n) \to \mathcal{S}(u, u, v)$."

Lemma 1.11 [18]: "Let $(\mathcal{X}, \mathcal{S})$ be an S-metric space and $\{u_n\}$ is a convergent sequence in $\mathcal{X}$. Then $\lim_{n\to\infty} u_n$ is unique."

Definition 1.12 [18]: "Let $(\mathcal{X}, \mathcal{S})$ be S-metric pace. A map $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ is said to be contraction if there exists a constant $k \in [0,1)$ such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq \lambda\, \mathcal{S}(u, u, v), \text{ for all } u, v \in \mathcal{X}."$$

Lemma 1.13 [18]: "If $\{u_n\}$ is a sequence of elements from S-metric space $(\mathcal{X}, \mathcal{S})$ satisfying the following property $\mathcal{S}(u_n, u_n, u_{n+1}) \leq k\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n)$, for each $k \epsilon$ [0, 1) where $n \epsilon \mathbb{N}$, then $\{u_n\}$ is a Cauchy sequence."

Finally, in this article we will apply the property introduced by G.S. Jeong and B.E. Rhoades in [22], [23] which they called the property P in metric spaces.

Definition 1.14: [22] "Let S be a self-mapping of S-metric space (X, S) with a nonempty fixed point set $F(T)$. Then T is said to satisfy the property P if $F(T) = F(T^n)$ for each $n \epsilon \mathbb{N}$."

2 Main Results

In this section, we establish fixed points of rational type contractions in the context of S-metric spaces and demonstrates that the P property is T-stable for some mappings. In order to show the relevance of the conclusions drawn in this work, we also provide a few examples.

Theorem 2.1: Let $(\mathcal{X}, \mathcal{S})$ be a complete S-metric space and $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a mapping such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq a_1\, \mathcal{S}(u, u, v) + a_2\, \frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}u) + \mathcal{S}(v,v,\mathcal{T}v)\mathcal{S}(u,u,\mathcal{T}v)}{\mathcal{S}(u,u,\mathcal{T}v) + \mathcal{S}(v,v,\mathcal{T}u)}, \tag{2.1}$$

for all $u, v \in \mathcal{X}$ and $a_1, a_2 \geq 0$, $\mathcal{S}(u, u, \mathcal{T}v) + \mathcal{S}(v, v, \mathcal{T}u) \neq 0$ with $a_1 + a_2 < 1$. Then, $\mathcal{T}$ has a unique fixed point $\mathcal{X}$.

Proof: Let u_0 be an arbitrary in $\mathcal{X}$, we define a sequence $\{u_n\}$ in $\mathcal{X}$ such that $\mathcal{T}u_n = u_{n+1}$ for all $n = 1,2, \ldots$. From condition (2.1) with $u = u_n$ and $v = u_{n-1}$, Therefore

$$\mathcal{S}(u_n, u_n, u_{n+1}) = \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n)$$

$$\leq a_1\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n)$$

$$+a_2\, \frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1}) + \mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n) + \mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})}$$

$$\leq a_1\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2\, \frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_n,u_n,u_n) + \mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_{n+1}) + \mathcal{S}(u_n,u_n,u_n)}$$

$$\leq a_1\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2\, \mathcal{S}(u_n, u_n, u_{n+1}).$$

It follows that

$$(1 - a_2)\mathcal{S}(u_n, u_n, u_{n+1}) \leq a_1\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \tag{2.2}$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{a_1}{1 - a_2}\right) \mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put $\lambda = \left(\frac{a_1}{1-a_2}\right)$. In view of $a_1 + a_2 < 1$, then $0 \leq \lambda < 1$. Thus, by Lemma 1.13, $\{u_n\}$ is a Cauchy sequence in $\mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

By (2.2), it is easy to see that

$$\mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) = \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*) \tag{2.3}$$

$$\leq a_1\, \mathcal{S}(u_n, u_n, u^*) + a_2\, \frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u^*,u^*,\mathcal{T}u_n)+\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u_n,u_n,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,\mathcal{T}u^*)+\mathcal{S}(u^*,u^*,\mathcal{T}u_n)}$$

$$\leq a_1 \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u^*,u^*,u_{n+1})+\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u_n,u_n,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,\mathcal{T}u^*)+\mathcal{S}(u^*,u^*,u_{n+1})}. \tag{2.4}$$

Taking the limit as $n \to \infty$ on both side of (2.4), we have $\lim_{n\to\infty} \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) = 0$.

That is, $u_n \to \mathcal{T}u^*$. Hence, $u^* = u^*$, u^* is a fixed point of $\mathcal{T}$.

Finally, we prove the uniqueness of the fixed point. Indeed, if there is another fixed point v^*, then by (2.1), we have

$$\mathcal{S}(u^*, u^*, v^*) = \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*)$$

$$\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2\, \frac{\mathcal{S}(u^*, u^*, \mathcal{T}u^*)\mathcal{S}(v^*, v^*, \mathcal{T}u^*) + \mathcal{S}(v^*, v^*, \mathcal{T}v^*)\mathcal{S}(u^*, u^*, \mathcal{T}v^*)}{\mathcal{S}(u^*, u^*, \mathcal{T}v^*) + \mathcal{S}(v^*, v^*, \mathcal{T}u^*)}$$

$$\leq a_1 \mathcal{S}(u^*, u^*, v^*) + a_2\, \frac{\mathcal{S}(u^*, u^*, u^*)\mathcal{S}(v^*, v^*, u^*) + \mathcal{S}(v^*, v^*, v^*)\mathcal{S}(u^*, u^*, v^*)}{\mathcal{S}(u^*, u^*, v^*) + \mathcal{S}(v^*, v^*, u^*)}$$

$$\mathcal{S}(u^*, u^*, v^*) \leq a_1 \mathcal{S}(u^*, u^*, v^*). \tag{2.5}$$

Since $a_1 + a_2 < 1$ implies $a_1 < 1$.

Therefore, we obtain that $\mathcal{S}(u^*, u^*, v^*) = 0$, i.e., $u^* = v^*$.

Hence the fixed point is unique.

This completes the proof. ■

Example 2.2: Let $\mathcal{X} = [0.1]$ be equipped with complete S-metric space define by

$$\mathcal{S}(u, v, w) = (|u - v| + |u - w| + |v - w|)^2.$$

Consider a mapping $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{T}(u) = \frac{1}{36} u^2 e^{-u^2},$$

for all $u, v, w \in \mathcal{X}$.

$$\begin{aligned}\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) &= (|\mathcal{T}u - \mathcal{T}u| + |\mathcal{T}u - \mathcal{T}w| + |\mathcal{T}u - \mathcal{T}w|)^2 \\ &= (2\, |\mathcal{T}u - \mathcal{T}w|)^2 \\ &= 4\left|\frac{1}{36}u^2e^{-u^2} - \frac{1}{36}w^2e^{-w^2}\right|^2 = \left|\frac{1}{18}u^2e^{-u^2} - \frac{1}{18}w^2e^{-w^2}\right|^2 \\ &\leq \frac{1}{9}\left|u^2e^{-u^2} - w^2e^{-w^2}\right|^2 \\ &\leq \frac{4}{9}|u - v|^2 = \frac{1}{9}|2(u - v)|^2 \\ &\leq \frac{1}{3}\mathcal{S}(u, v, w) \\ &\leq a_1\, \mathcal{S}(u, u, v) + a_2\, \frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}u)+\mathcal{S}(v,v,\mathcal{T}v)\mathcal{S}(u,u,\mathcal{T}v)}{\mathcal{S}(u,u,\mathcal{T}v)+ \mathcal{S}(v,v,\mathcal{T}u)}.\end{aligned}$$

Clearly by taking $a_2 = \frac{1}{2}$, we have $a_1 + a_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} < 1$. Then, from Theorem 2.1 we conclude that, $\mathcal{T}$ has a unique fixed point. Also, 0 is the only fixed point of $\mathcal{T}$.

Theorem 2.3: Let $(\mathcal{X}, \mathcal{S})$ be a complete S-metric space and $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a mapping such that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}v) \leq a_1\, \mathcal{S}(u, u, v) + a_2 \frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(u,u,\mathcal{T}v) + \mathcal{S}(v,v,\mathcal{T}v)\mathcal{S}(v,v,\mathcal{T}u)}{\mathcal{S}(u,u,\mathcal{T}v) + \mathcal{S}(v,v,\mathcal{T}u)}$$

$$+a_3 \frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}u) + \mathcal{S}(v,v,\mathcal{T}v)\mathcal{S}(u,u,\mathcal{T}v)}{\mathcal{S}(u,u,\mathcal{T}v) + \mathcal{S}(v,v,\mathcal{T}u)}, \quad (2.6)$$

where a_1, a_2, a_3 are non-negative constant with $a_1 + a_2 + a_3 < 1$. Then, $\mathcal{T}$ has a unique fixed point $\mathcal{X}$.

Proof: Choose $u_0 \in X$ and construct a Picard iterative sequence $\{u_n\}$ as $\mathcal{T}u_n = u_{n+1}$. If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, then $u_{n_0} = u_{n_0+1} = \mathcal{T}u_{n_0}$, i.e., u_{n_0} is a fixed point of $\mathcal{T}$. Next, without loss of generality, let $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$, Using (2.6), we get

$$\begin{aligned}
\mathcal{S}(u_n, u_n, u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n) \\
&\leq a_1\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \\
&+a_2 \frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n) + \mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n) + \mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})} \\
&+ a_3 \frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1}) + \mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n) + \mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})} \\
&\leq a_1\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) \\
&+ a_2 \frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_{n-1},u_{n-1},u_{n+1}) + \mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u_n,u_n,u_n)}{\mathcal{S}(u_{n-1},u_{n-1},u_{n+1}) + \mathcal{S}(u_n,u_n,u_n)} \\
&+ a_3 \frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_n,u_n,u_n) + \mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_{n+1}) + \mathcal{S}(u_n,u_n,u_n)} \\
&\leq a_1\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_3\, \mathcal{S}(u_n, u_n, u_{n+1}).
\end{aligned}$$

It follows that

$$(1 - a_3)\mathcal{S}(u_n, u_n, u_{n+1}) \leq (a_1 + a_2)\mathcal{S}(u_{n-1}, u_{n-1}, u_n) \quad (2.7)$$

$$\mathcal{S}(u_n, u_n, u_{n+1}) \leq \left(\frac{a_1 + a_2}{1 - a_3}\right)\mathcal{S}(u_{n-1}, u_{n-1}, u_n).$$

Put $\lambda = \frac{a_1 + a_2}{1 - a_3}$. In view of $a_1 + a_2 + a_3 < 1$, we have $0 \leq \lambda < 1$. Thus, from Lemma 1.13 $\{u_n\}$ is Cauchy sequence in $\mathcal{X}$. Since, $(\mathcal{X}, \mathcal{S})$ is a complete S-metric space, so there exists some point $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

Again from (2.6) it is easy to see that

$$\begin{aligned}
\mathcal{S}(u^*, u^*, \mathcal{T}u^*) &\leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*) \quad (2.8) \\
&\leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*) \\
&\leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + a_1\, \mathcal{S}(u_n, u_n, u^*) \\
&+ a_2 \frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u_n,u_n,\mathcal{T}u^*) + \mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,\mathcal{T}u_n)}{\mathcal{S}(u_n,u_n,\mathcal{T}u^*) + \mathcal{S}(u^*,u^*,\mathcal{T}u_n)} \\
&+ a_3 \frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u^*,u^*,\mathcal{T}u_n) + \mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u_n,u_n,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,\mathcal{T}u^*) + \mathcal{S}(u^*,u^*,\mathcal{T}u_n)} \\
&\leq 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + a_1\, \mathcal{S}(u_n, u_n, u^*) \\
&+a_2 \frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u_n,u_n,\mathcal{T}u^*) + \mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,u_{n+1})}{\mathcal{S}(u_n,u_n,\mathcal{T}u^*) + \mathcal{S}(u^*,u^*,u_{n+1})} \\
&+a_3 \frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u^*,u^*,u_{n+1}) + \mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u_n,u_n,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,\mathcal{T}u^*) + \mathcal{S}(u^*,u^*,u_{n+1})}. \quad (2.9)
\end{aligned}$$

Taking the limit as $n \to \infty$ on both side of (2.9), we have $\lim_{n\to\infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$.

Hence, $\mathcal{T}u^* = u^*$ it follows that u^* is a fixed point of $\mathcal{T}$.

Next, we claim the uniqueness of fixed point.

Indeed, if there is another fixed point v^*, then by (2.6), we have

$$\begin{aligned}\mathcal{S}(u^*,u^*,v^*) &= \mathcal{S}(\mathcal{T}u^*,\mathcal{T}u^*,\mathcal{T}v^*)\\ &\leq a_1\,\mathcal{S}(u^*,u^*,v^*) + a_2\frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,\mathcal{T}v^*)+\mathcal{S}(v^*,v^*,\mathcal{T}v^*)\mathcal{S}(v^*,v^*,\mathcal{T}u^*)}{\mathcal{S}(u^*,u^*,\mathcal{T}v^*)+\mathcal{S}(v^*,v^*,\mathcal{T}u^*)}\\ &+ a_3\frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(v^*,v^*,\mathcal{T}u^*)+\mathcal{S}(v^*,v^*,\mathcal{T}v^*)\mathcal{S}(u^*,u^*,\mathcal{T}v^*)}{\mathcal{S}(u^*,u^*,\mathcal{T}v^*)+\mathcal{S}(v^*,v^*,\mathcal{T}u^*)}\\ &\leq a_1\,\mathcal{S}(u^*,u^*,v^*) + a_2\frac{\mathcal{S}(u^*,u^*,u^*)\mathcal{S}(u^*,u^*,v^*)+\mathcal{S}(v^*,v^*,v^*)\mathcal{S}(v^*,v^*,u^*)}{\mathcal{S}(u^*,u^*,v^*)+\mathcal{S}(v^*,v^*,u^*)}\\ &+a_3\frac{\mathcal{S}(u^*,u^*,u^*)\mathcal{S}(v^*,v^*,u^*)+\mathcal{S}(v^*,v^*,v^*)\mathcal{S}(u^*,u^*,v^*)}{\mathcal{S}(u^*,u^*,v^*)+\mathcal{S}(v^*,v^*,u^*)}\end{aligned}$$

$$\mathcal{S}(u^*,u^*,v^*) \leq a_1\mathcal{S}(u^*,u^*,v^*). \tag{2.10}$$

Since $a_1 + a_2 + a_3 < 1 \Rightarrow a_1 < 1$, we obtain that $\mathcal{S}(u^*,u^*,v^*) = 0$, i.e., $u^* = v^*$.

Hence the fixed point is unique.

This completes the proof. ■

Theorem 2.4: Let $(\mathcal{X},\mathcal{S})$ be a complete S-metric space. Let $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ be a mapping satisfying

$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) \leq a_1\,\mathcal{S}(u,u,v) + a_2\,\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)} + a_3\,\frac{\mathcal{S}(v,v,\mathcal{T}v)[1+\mathcal{S}(u,u,\mathcal{T}u)]}{1+\mathcal{S}(u,u,v)}, \tag{2.11}$$

for all $u, v \in \mathcal{X}$ and a_1, a_2, a_3 are non-negative constant with $a_1 + a_2 + a_3 < 1$. Then $\mathcal{T}$ has a unique fixed point $\mathcal{X}$.

Proof: Choose $u_0 \in X$. Construct a sequence $\{u_n\}$ in $\mathcal{X}$ by $\mathcal{T}u_n = u_{n+1}$. For all $n \in \mathbb{N}$, from condition (2.11) with $u = u_n$ and $v = u_{n-1}$, we have

$$\begin{aligned}\mathcal{S}(u_n,u_n,u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n)\\ &\leq a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_2\,\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_n,u_n,\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}\\ &+a_3\,\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)[1+\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})}{1+\mathcal{S}(u_{n-1},u_{n-1},u_n)}\\ &\leq a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_2\,\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_n,u_n,u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}\\ &+a_3\,\frac{\mathcal{S}(u_n,u_n,u_{n+1})[1+\mathcal{S}(u_{n-1},u_{n-1},u_n)]}{1+\mathcal{S}(u_{n-1},u_{n-1},u_n)}\\ &\leq a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_2\,\mathcal{S}(u_n,u_n,u_{n+1}) + a_3\,\mathcal{S}(u_n,u_n,u_{n+1}).\end{aligned}$$

It follows that

$$(1 - a_2 - a_3)\mathcal{S}(u_n,u_n,u_{n+1}) \leq a_1\mathcal{S}(u_{n-1},u_{n-1},u_n) \tag{2.12}$$

$$\mathcal{S}(u_n,u_n,u_{n+1}) \leq \left(\frac{a_1}{1-a_2-a_3}\right)\mathcal{S}(u_{n-1},u_{n-1},u_n).$$

Put $\lambda = \frac{a_1}{1-a_2-a_3}$ .In view of $a_1 + a_2 + a_3 < 1$, we have $0 \leq \lambda < 1$. Thus, from Lemma 1.13 $\{u_n\}$ is Cauchy sequence in $\mathcal{X}$. Since, $(\mathcal{X},\mathcal{S})$ is a complete S-metric space, so there exists some point $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

Again from (2.11) it is easy to see that

$$\begin{aligned}\mathcal{S}(u^*,u^*,\mathcal{T}u^*) &\leq 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + \mathcal{S}(u_{n+1},u_{n+1},\mathcal{T}u^*)\\ &\leq 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + \mathcal{S}(\mathcal{T}u_n,\mathcal{T}u_n,\mathcal{T}u^*)\\ &\leq 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + a_1\,\mathcal{S}(u_n,u_n,u^*) + a_2\,\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u^*,u^*,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,u^*)}\end{aligned} \tag{2.13}$$

$$+a_3 \frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)[1+\mathcal{S}(u_n,u_n,\mathcal{T}u_n)]}{1+\mathcal{S}(u_n,u_n,u^*)}$$
$$\leq 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + a_1\,\mathcal{S}(u_n,u_n,u^*) + a_2 \frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u^*,u^*,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,u^*)}$$
$$+ a_3 \frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)[1+\mathcal{S}(u_n,u_n,\mathcal{T}u_{n+1})]}{1+\mathcal{S}(u_n,u_n,u^*)}. \quad (2.14)$$

Taking the limit as $n \to \infty$ on both side of (2.14), we have $\lim_{n\to\infty} \mathcal{S}(u^*,u^*,\mathcal{T}u^*) = 0$.

Hence, $\mathcal{T}u^* = u^*$ it follows that u^* is a fixed point of $\mathcal{T}$.

Next, we claim the uniqueness of fixed point.

Indeed, if there is another fixed point v^*, then by (2.11), we have

$$\mathcal{S}(u^*,u^*,v^*) = \mathcal{S}(\mathcal{T}u^*,\mathcal{T}u^*,\mathcal{T}v^*)$$
$$\leq a_1\,\mathcal{S}(u^*,u^*,v^*) + a_1 \frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(v^*,v^*,\mathcal{T}v^*)}{\mathcal{S}(u^*,u^*,v^*)} + a_1 \frac{\mathcal{S}(v^*,v^*,\mathcal{T}v^*)[1+\mathcal{S}(u^*,u^*,\mathcal{T}u^*)]}{1+\mathcal{S}(u^*,u^*,v^*)}$$
$$\leq a_1\,\mathcal{S}(u^*,u^*,v^*) + a_2 \frac{\mathcal{S}(u^*,u^*,u^*)\mathcal{S}(v^*,v^*,v^*)}{\mathcal{S}(u^*,u^*,v^*)} + a_3 \frac{\mathcal{S}(v^*,v^*,v^*)[1+\mathcal{S}(u^*,u^*,u^*)]}{1+\mathcal{S}(u^*,u^*,v^*)}$$
$$\mathcal{S}(u^*,u^*,v^*) \leq a_1\,\mathcal{S}(u^*,u^*,v^*).$$

Since $0 < a_1 + a_2 + a_3 < 1 \Rightarrow a_1 < 1$, thus, we obtain $\mathcal{S}(u^*,u^*,v^*) = 0$, i.e., $u^* = v^*$.

Hence, we proved that $\mathcal{T}$ have a unique fixed point in $\mathcal{X}$.

Here completes the proof. ■

Example 2.5: Let $\mathcal{X} = [0.1]$ and $(\mathcal{X},\mathcal{S})$ be a usual S-metric space which is complete, define by

$$\mathcal{S}(u,v,w) = |u-w| + |v-w|.$$

Consider a mapping $\mathcal{T}: X \to X$ be define as $(u) = \frac{u}{8}$, for all $u,v,w \in X$. Obviously,

$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) = 2\,|\mathcal{T}u - \mathcal{T}v| = 2\,\left|\frac{u}{8} - \frac{v}{8}\right| = \frac{1}{4}\,|u-v|,$$
$$\mathcal{S}(u,u,v) = 2\,|u-v|.$$

Also,

$$\mathcal{S}(u,u,\mathcal{T}u) = 2\,|u-\mathcal{T}u| = 2\,\left|u - \frac{u}{8}\right| = \frac{7u}{4},$$
$$\mathcal{S}(v,v,\mathcal{T}v) = 2\,|v-\mathcal{T}v| = 2\,\left|v - \frac{v}{8}\right| = \frac{7v}{4},$$
$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) = \frac{1}{4}\,|u-v|$$
$$= \frac{1}{8}\,2\,|u-v|$$
$$\leq \frac{1}{8}\,\mathcal{S}(u,u,v) + \frac{1}{4} \frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)} + \frac{1}{7} \frac{\mathcal{S}(v,v,\mathcal{T}v)[1+\mathcal{S}(u,u,\mathcal{T}u)]}{1+\mathcal{S}(u,u,v)}.$$

It is clear that, $a_1 + a_2 + a_3 = \frac{1}{8} + \frac{1}{4} + \frac{1}{7} = \frac{29}{56} < 1$. Thus, we conclude that inequality (2.11) of Theorem 2.4 remains valid. Hence, $\mathcal{T}$ has a unique fixed point and the fixed point is 0.

Theorem 2.6: Let $(\mathcal{X},\mathcal{S})$ be a complete S-metric space and $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a mapping satisfying the following condition:

$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) \leq a_1\,\mathcal{S}(u,u,v) + a_2 \frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)} + a_3 \frac{\mathcal{S}(u,u,\mathcal{T}v)\mathcal{S}(v,v,\mathcal{T}u)}{\mathcal{S}(u,u,v)}$$
$$+ a_4\,[\mathcal{S}(u,u,\mathcal{T}u) + \mathcal{S}(v,v,\mathcal{T}v)] + a_5\,[\mathcal{S}(v,v,\mathcal{T}u) + \mathcal{S}(u,u,\mathcal{T}v)], \quad (2.15)$$

for all $u, v \in \mathcal{X}$ and a_1, a_2, a_3, a_4, a_5 are non-negative constant $a_1 + a_2 + a_3 + 2\, a_4 + 3\, a_5 < 1$. Then, $\mathcal{T}$ has a unique fixed point $\mathcal{X}$.

Proof: Choose $u_0 \in X$. Construct a sequence $\{u_n\}$ in $\mathcal{X}$ by $\mathcal{T}u_n = u_{n+1}$.

For all $n \in \mathbb{N}$, from condition (2.15) with $u = u_n$ and $v = u_{n-1}$, we have

$$\begin{aligned}\mathcal{S}(u_n, u_n, u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1}, \mathcal{T}u_{n-1}, \mathcal{T}u_n)\\ &\le a_1\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1})\mathcal{S}(u_n, u_n, \mathcal{T}u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)}\\ &+a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n)\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)}\\ &+a_4[\mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_{n-1}) + \mathcal{S}(u_n, u_n, \mathcal{T}u_n)]\\ &+a_5[\mathcal{S}(u_n, u_n, \mathcal{T}u_{n-1}) + \mathcal{S}(u_{n-1}, u_{n-1}, \mathcal{T}u_n)]\\ &\le a_1\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)\mathcal{S}(u_n, u_n, u_{n+1})}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)}\\ &+a_3 \frac{\mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1})\mathcal{S}(u_n, u_n, u_n)}{\mathcal{S}(u_{n-1}, u_{n-1}, u_n)} + a_4[\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})]\\ &+a_5[\mathcal{S}(u_n, u_n, u_n) + \mathcal{S}(u_{n-1}, u_{n-1}, u_{n+1})]\\ &\le a_1\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + a_2\, \mathcal{S}(u_n, u_n, u_{n+1})\\ &+a_4[\mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})]\\ &+a_5[2\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n) + \mathcal{S}(u_n, u_n, u_{n+1})].\end{aligned}$$

It follows that:

$$\begin{aligned}(1 - a_2 - a_4 - a_5)\mathcal{S}(u_n, u_n, u_{n+1}) &\le (a_1 + a_4 + 2a_5)\, \mathcal{S}(u_{n-1}, u_{n-1}, u_n)\\ \mathcal{S}(u_n, u_n, u_{n+1}) &\le \left(\frac{a_1 + a_4 + 2a_5}{1 - a_2 - a_4 - a_5}\right)\mathcal{S}(u_{n-1}, u_{n-1}, u_n).\end{aligned} \tag{2.16}$$

Put $\lambda = \frac{a_1 + a_4 + 2a_5}{1 - a_2 - a_4 - a_5}$. In view of $a_1 + a_2 + a_3 + 2a_4 + 3a_5 < 1$, we have $0 \le \lambda < 1$. Thus, from Lemma 1.13, $\{u_n\}$ is Cauchy sequence in $\mathcal{X}$. Since, $(\mathcal{X}, \mathcal{S})$ is a complete S-metric space, so there exists some point $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

Again from (2.15) it is easy to see that:

$$\begin{aligned}\mathcal{S}(u^*, u^*, \mathcal{T}u^*) &\le 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_{n+1}, u_{n+1}, \mathcal{T}u^*)\\ &\le 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(\mathcal{T}u_n, \mathcal{T}u_n, \mathcal{T}u^*)\\ &\le 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + a_1\, \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u_n)\mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)}\\ &+a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, \mathcal{T}u_n)}{\mathcal{S}(u_n, u_n, u^*)} + a_4[\mathcal{S}(u_n, u_n, \mathcal{T}u_n) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)]\\ &+a_5[\mathcal{S}(u^*, u^*, \mathcal{T}u_n) + \mathcal{S}(u_n, u_n, \mathcal{T}u^*)]\\ &\le 2\, \mathcal{S}(u^*, u^*, u_{n+1}) + a_1\, \mathcal{S}(u_n, u_n, u^*) + a_2 \frac{\mathcal{S}(u_n, u_n, u_{n+1})\mathcal{S}(u^*, u^*, \mathcal{T}u^*)}{\mathcal{S}(u_n, u_n, u^*)}\\ &+ a_3 \frac{\mathcal{S}(u_n, u_n, \mathcal{T}u^*)\mathcal{S}(u^*, u^*, u_{n+1})}{1 + \mathcal{S}(u_n, u_n, u^*)} + a_4[\mathcal{S}(u_n, u_n, u_{n+1}) + \mathcal{S}(u^*, u^*, \mathcal{T}u^*)]\\ &+a_5[\mathcal{S}(u^*, u^*, u_{n+1}) + \mathcal{S}(u_n, u_n, \mathcal{T}u^*)\end{aligned} \tag{2.17}$$

(2.18)

Taking the limit as $n \to \infty$ on both side of (2.18), we have $\lim_{n \to \infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$.

Hence, $\mathcal{T}u^* = u^*$ it follows that u^* is a fixed point of $\mathcal{T}$.

Finally, we prove the uniqueness of fixed point.

Indeed, if there is another fixed point v^*, then by (2.15), we have

$$\begin{aligned}\mathcal{S}(u^*,u^*,v^*) &= \mathcal{S}(\mathcal{T}u^*,\mathcal{T}u^*,\mathcal{T}v^*)\\ &\le a_1\,\mathcal{S}(u^*,u^*,v^*) + a_2\,\frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(v^*,v^*,\mathcal{T}v^*)}{\mathcal{S}(u^*,u^*,v^*)} + a_3\,\frac{\mathcal{S}(u^*,u^*,\mathcal{T}v^*)\mathcal{S}(v^*,v^*,\mathcal{T}u^*)}{\mathcal{S}(u^*,u^*,v^*)}\\ &\quad + a_4\,[\mathcal{S}(u^*,u^*,\mathcal{T}u^*) + \mathcal{S}(v^*,v^*,\mathcal{T}v^*)] + a_5[\mathcal{S}(v^*,v^*,\mathcal{T}u^*) + \mathcal{S}(u^*,u^*,\mathcal{T}v^*)]\\ &\le a_1\,\mathcal{S}(u^*,u^*,v^*) + a_2\,\frac{\mathcal{S}(u^*,u^*,u^*)\mathcal{S}(v^*,v^*,v^*)}{\mathcal{S}(u^*,u^*,v^*)} + a_3\,\frac{\mathcal{S}(u^*,u^*,v^*)\mathcal{S}(v^*,v^*,u^*)}{\mathcal{S}(u^*,u^*,v^*)}\\ &\quad + a_4\,[\mathcal{S}(u^*,u^*,u^*) + \mathcal{S}(v^*,v^*,v^*)] + a_5[\mathcal{S}(v^*,v^*,u^*) + \mathcal{S}(u^*,u^*,v^*)]\end{aligned}$$

$$\mathcal{S}(u^*,u^*,v^*) \le (a_1 + a_3 + 2a_5)\,\mathcal{S}(u^*,u^*,v^*). \tag{2.19}$$

Since $0 < a_1 + a_2 + a_3 + 2a_4 + 3a_5 < 1 \Rightarrow a_1 + a_3 + 2a_5 < 1$, thus, we obtain $\mathcal{S}(u^*,u^*,v^*) = 0$, which further implies $u^* = v^*$.

Therefore, $\mathcal{T}$ have a unique fixed point in $\mathcal{X}$.

Here completes the proof. ■

Theorem 2.7: Let $(\mathcal{X},\mathcal{S})$ be a complete S-metric space and $\mathcal{T}:\mathcal{X} \to \mathcal{X}$ be a self map that satisfies the following inequality

$$\begin{aligned}\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) &\le a_1\,\mathcal{S}(u,u,v) + a_2\,\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)} + a_3\,\frac{\mathcal{S}(u,u,\mathcal{T}v)\mathcal{S}(v,v,\mathcal{T}u)}{\mathcal{S}(u,u,v)}\\ &\quad + a_4\,\frac{\mathcal{S}(v,v,\mathcal{T}v)[1+\mathcal{S}(u,u,\mathcal{T}u)]}{1+\mathcal{S}(u,u,v)},\end{aligned} \tag{2.20}$$

for all $u,v \in \mathcal{X}$ and a_1,a_2,a_3,a_4 are non-negative constant $a_1 + a_2 + a_3 + a_4 < 1$. Then, $\mathcal{T}$ has a unique fixed point $\mathcal{X}$.

Proof: Choose $u_0 \in X$ and construct a Picard iterative sequence $\{u_n\}$ as $\mathcal{T}u_n = u_{n+1}$.

For all $n \in \mathbb{N}$, from condition (2.20) with $u = u_n$ and $v = u_{n-1}$, we have

$$\begin{aligned}\mathcal{S}(u_n,u_n,u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n)\\ &\le a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_2\,\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_n,u_n,\mathcal{T}u_n)}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}\\ &\quad + a_3\,\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}\\ &\quad + a_4\,\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)[1+\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})]}{1+\mathcal{S}(u_{n-1},u_{n-1},u_n)}\\ &\le a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_2\,\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_n,u_n,u_{n+1})}{\mathcal{S}(u_{n-1},u_{n-1},u_n)}\\ &\quad + a_3\,\frac{\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})\mathcal{S}(u_n,u_n,u_n)}{\mathcal{S}(u_{n-1},u_{n-1},u_n)} + a_4\,\frac{\mathcal{S}(u_n,u_n,u_{n+1})[1+\mathcal{S}(u_{n-1},u_{n-1},u_n)]}{1+\mathcal{S}(u_{n-1},u_{n-1},u_n)}\\ &\le a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_2\,\mathcal{S}(u_n,u_n,u_{n+1}) + a_4\mathcal{S}(u_n,u_n,u_{n+1}),\end{aligned}$$

which further implies,

$$\begin{aligned}(1 - a_2 - a_4)\mathcal{S}(u_n,u_n,u_{n+1}) &\le a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n)\\ \mathcal{S}(u_n,u_n,u_{n+1}) &\le \left(\frac{a_1}{1-a_2-a_4}\right)\mathcal{S}(u_{n-1},u_{n-1},u_n).\end{aligned} \tag{2.21}$$

Put $\lambda = \frac{a_1}{1-a_2-a_4}$. In view of $a_1 + a_2 + a_3 + a_4 < 1$, we have $0 \le \lambda < 1$. Thus, from Lemma 1.13, $\{u_n\}$ is Cauchy sequence in $\mathcal{X}$. Since, $(\mathcal{X},\mathcal{S})$ is a complete S-metric space, so there exists some point $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

Again from (2.20) it is easy to see that:

$$\begin{aligned}\mathcal{S}(u^*,u^*,\mathcal{T}u^*) &\le 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + \mathcal{S}(u_{n+1},u_{n+1},\mathcal{T}u^*) \qquad (2.22)\\ &= 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + \mathcal{S}(\mathcal{T}u_n,\mathcal{T}u_n,\mathcal{T}u^*)\\ &\le 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + a_1\,\mathcal{S}(u_n,u_n,u^*) + a_2\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u^*,u^*,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,u^*)}\\ &+a_3\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,\mathcal{T}u_n)}{\mathcal{S}(u_n,u_n,u^*)} + a_4\frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)[1+\mathcal{S}(u_n,u_n,\mathcal{T}u_n)]}{1+\mathcal{S}(u_n,u_n,u^*)}\\ &\le 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + a_1\,\mathcal{S}(u_n,u_n,u^*) + a_2\frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u^*,u^*,\mathcal{T}u^*)}{\mathcal{S}(u_n,u_n,u^*)}\\ &+ a_3\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,u_{n+1})}{1+\mathcal{S}(u_n,u_n,u^*)} + a_4\frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)[1+\mathcal{S}(u_n,u_n,u_{n+1})]}{1+\mathcal{S}(u_n,u_n,u^*)}. \qquad (2.23)\end{aligned}$$

Taking the limit as $n \to \infty$ on both side of (2.23), we have $\lim_{n\to\infty} \mathcal{S}(u^*,u^*,\mathcal{T}u^*) = 0.$

Hence, $\mathcal{T}u^* = u^*$ it follows that u^* is a fixed point of $\mathcal{T}$.

Finally, we prove the uniqueness of fixed point.

Indeed, if there is another fixed point v^*, then by (2.20), we have

$$\begin{aligned}\mathcal{S}(u^*,u^*,v^*) &= \mathcal{S}(\mathcal{T}u^*,\mathcal{T}u^*,\mathcal{T}v^*)\\ &\le a_1\,\mathcal{S}(u^*,u^*,v^*) + a_1\,\frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(v^*,v^*,\mathcal{T}v^*)}{\mathcal{S}(u^*,u^*,v^*)} + a_1\,\frac{\mathcal{S}(u^*,u^*,\mathcal{T}v^*)\mathcal{S}(v^*,v^*,\mathcal{T}u^*)}{\mathcal{S}(u^*,u^*,v^*)}\\ &+a_4\,\frac{\mathcal{S}(v^*,v^*,\mathcal{T}v^*)[1+\mathcal{S}(u^*,u^*,\mathcal{T}u^*)]}{1+\mathcal{S}(u^*,u^*,v^*)}\\ &\le a_1\,\mathcal{S}(u^*,u^*,v^*) + a_2\,\frac{\mathcal{S}(u^*,u^*,u^*)\mathcal{S}(v^*,v^*,v^*)}{\mathcal{S}(u^*,u^*,v^*)} + a_3\,\frac{\mathcal{S}(u^*,u^*,v^*)\mathcal{S}(v^*,v^*,u^*)}{\mathcal{S}(u^*,u^*,v^*)}\\ &+a_4\,\frac{\mathcal{S}(v^*,v^*,v^*)[1+\mathcal{S}(u^*,u^*,u^*)]}{1+\mathcal{S}(u^*,u^*,v^*)}.\\ \mathcal{S}(u^*,u^*,v^*) &\le (a_1+a_3)\,\mathcal{S}(u^*,u^*,v^*)\\ \mathcal{S}(u^*,u^*,v^*) &\le (a_1+a_2+a_3+a_4)\,\mathcal{S}(u^*,u^*,v^*), \qquad (2.24)\end{aligned}$$

a contradiction.

Thus, we obtain $\mathcal{S}(u^*,u^*,v^*) = 0$, which further implies $u^* = v^*$.
Therefore, $\mathcal{T}$ have a unique fixed point in $\mathcal{X}$

Here completes the proof. ■

Example 2.8: Let $\mathcal{X} = [0.1]$ be equipped with complete S-metric space define by

$$\mathcal{S}(u,v,w) = (|u-v| + |u-w| + |v-w|)^2.$$

Let the mapping $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be defined by

$$\mathcal{T}(u) = \frac{u}{5}.$$

Then, for all $u, v, w \in \mathcal{X}$, we have,

$$\begin{aligned}&\mathcal{S}(u,u,v) = 4|u-v|^2\ ,\\ &\mathcal{S}(u,u,\mathcal{T}u) = 4|u-\mathcal{T}u|^2 = 4\left|u-\frac{u}{5}\right|^2 = \frac{16}{25}u^2\\ &\mathcal{S}(v,v,\mathcal{T}v) = 4|v-\mathcal{T}v|^2 = 4\left|v-\frac{v}{5}\right|^2 = \frac{16}{25}v^2\end{aligned}$$

Also,

$$\begin{aligned}\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) &= (|\mathcal{T}u-\mathcal{T}u| + |\mathcal{T}u-\mathcal{T}w| + |\mathcal{T}u-\mathcal{T}w|)^2 \\ &= (2\,|\mathcal{T}u-\mathcal{T}w|)^2 \\ &= \frac{4}{25}|u-v|^2 \\ &\le \frac{1}{25}\mathcal{S}(u,v,w) + \frac{4}{25}\,\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(v,v,\mathcal{T}v)}{\mathcal{S}(u,u,v)} + \frac{2}{5}\,\frac{\mathcal{S}(v,v,\mathcal{T}v)[1+\mathcal{S}(u,u,\mathcal{T}u)]}{1+\mathcal{S}(u,u,v)}\end{aligned}$$

Clearly, we have $a_1 + a_2 + a_3 = \frac{1}{25} + \frac{4}{25} + \frac{2}{5} = \frac{3}{5} < 1$. Then, from Theorem 2.7 we conclude that, $\mathcal{T}$ has a unique fixed point. Also, 0 is the only fixed point of $\mathcal{T}$.

Theorem 2.9: Let $(\mathcal{X},\mathcal{S})$ be a complete S-metric space. Let $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a mapping satisfying

$$\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v) \le a_1\,\mathcal{S}(u,u,v) + a_2\,\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(u,u,\mathcal{T}v)+\mathcal{S}(v,v,\mathcal{T}v)\mathcal{S}(v,v,\mathcal{T}u)}{\mathcal{S}(u,u,\mathcal{T}v)+\mathcal{S}(v,v,\mathcal{T}u)} + a_3\,\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}v)\,, \qquad (2.25)$$

for all $u, v \in \mathcal{X}$ and $a_1, a_2, a_3 \ge 0$. $\mathcal{S}(u,u,\mathcal{T}v) + \mathcal{S}(v,v,\mathcal{T}u) \ne 0$ with $a_1 + a_2 + a_3 < 1$. Then $\mathcal{T}$ has a unique fixed point $\mathcal{X}$.

Proof: Choose u_0 as an arbitrary point in X. We define a sequence $\{u_n\}$ in $\mathcal{X}$ by $\mathcal{T}u_n = u_{n+1}$.Then or all $n \in \mathbb{N}$, from condition (2.25) with $u = u_n$ and $v = u_{n-1}$, we have

$$\begin{aligned}\mathcal{S}(u_n,u_n,u_{n+1}) &= \mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n) \\ &\le a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n) \\ &\quad + a_2\,\frac{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_{n-1})\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)+\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})}{\mathcal{S}(u_{n-1},u_{n-1},\mathcal{T}u_n)+\mathcal{S}(u_n,u_n,\mathcal{T}u_{n-1})} \\ &\quad + a_3\,\mathcal{S}(\mathcal{T}u_{n-1},\mathcal{T}u_{n-1},\mathcal{T}u_n) \\ &\le a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n) \\ &\quad + a_2\,\frac{\mathcal{S}(u_{n-1},u_{n-1},u_n)\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})+\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u_n,u_n,u_n)}{\mathcal{S}(u_{n-1},u_{n-1},u_{n+1})+\mathcal{S}(u_n,u_n,u_n)} \\ &\quad + a_3\mathcal{S}(u_n,u_n,u_{n+1}) \\ &\le a_1\,\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_2\mathcal{S}(u_{n-1},u_{n-1},u_n) + a_3\mathcal{S}(u_n,u_n,u_{n+1}).\end{aligned}$$

Which further implies

$$\begin{aligned}(1-a_3)\mathcal{S}(u_n,u_n,u_{n+1}) &\le (a_1+a_2)\mathcal{S}(u_{n-1},u_{n-1},u_n) \qquad (2.26) \\ \mathcal{S}(u_n,u_n,u_{n+1}) &\le \left(\frac{a_1+a_2}{1-a_3}\right)\mathcal{S}(u_{n-1},u_{n-1},u_n).\end{aligned}$$

Put $\lambda = \frac{a_1+a_2}{1-a_3}$. In view of $a_1 + a_2 + a_3 < 1$, we have $0 \le \lambda < 1$. Thus, from Lemma 1.13 $\{u_n\}$ is Cauchy sequence in $\mathcal{X}$. Since, $(\mathcal{X},\mathcal{S})$ is a complete S-metric space, so there exists some point $u^* \in \mathcal{X}$ such that $u_n \to u^*$ as $n \to \infty$.

Again from (2.25) it is easy to see that

$$\begin{aligned}\mathcal{S}(u^*,u^*,\mathcal{T}u^*) &\le 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + \mathcal{S}(u_{n+1},u_{n+1},\mathcal{T}u^*) \qquad (2.27) \\ &\le 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + \mathcal{S}(\mathcal{T}u_n,\mathcal{T}u_n,\mathcal{T}u^*) \\ &\le 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + a_1\,\mathcal{S}(u_n,u_n,u^*) \\ &\quad + a_2\,\frac{\mathcal{S}(u_n,u_n,\mathcal{T}u_n)\mathcal{S}(u_n,u_n,\mathcal{T}u^*)+\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,\mathcal{T}u_n)}{\mathcal{S}(u_n,u_n,\mathcal{T}u^*)+\mathcal{S}(u^*,u^*,\mathcal{T}u_n)} \\ &\quad + a_3\,\mathcal{S}(\mathcal{T}u_n,\mathcal{T}u_n,\mathcal{T}u^*) \\ &\le 2\,\mathcal{S}(u^*,u^*,u_{n+1}) + a_1\,\mathcal{S}(u_n,u_n,u^*) \\ &\quad + a_2\,\frac{\mathcal{S}(u_n,u_n,u_{n+1})\mathcal{S}(u_n,u_n,\mathcal{T}u^*)+\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,u_{n+1})}{\mathcal{S}(u_n,u_n,\mathcal{T}u^*)+\mathcal{S}(u^*,u^*,u_{n+1})} \\ &\quad + a_3\,\mathcal{S}(u_{n+1},u_{n+1},\mathcal{T}u^*)\,. \qquad (2.28)\end{aligned}$$

Taking the limit

as $n \to \infty$ on both side of (2.28), we have $\lim_{n\to\infty} \mathcal{S}(u^*, u^*, \mathcal{T}u^*) = 0$.

Hence, $\mathcal{T}u^* = u^*$ it follows that u^* is a fixed point of $\mathcal{T}$.

Finally, we claim the uniqueness of fixed point.

Indeed, if there is another fixed point v^*, then by (2.25), we have

$$\begin{aligned} \mathcal{S}(u^*, u^*, v^*) &= \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*) \\ &\leq a_1\, \mathcal{S}(u^*, u^*, v^*) + a_2\, \frac{\mathcal{S}(u^*,u^*,\mathcal{T}u^*)\mathcal{S}(u^*,u^*,\mathcal{T}v^*)+\mathcal{S}(v^*,v^*,\mathcal{T}v^*)\mathcal{S}(v^*,v^*,\mathcal{T}u^*)}{\mathcal{S}(u^*,u^*,\mathcal{T}v^*)+\mathcal{S}(v^*,v^*,\mathcal{T}u^*)} \\ &\quad + a_3\, \mathcal{S}(\mathcal{T}u^*, \mathcal{T}u^*, \mathcal{T}v^*) \\ &\leq a_1\, \mathcal{S}(u^*, u^*, v^*) + a_2\, \frac{\mathcal{S}(u^*,u^*,u^*)\mathcal{S}(u^*,u^*,v^*)+\mathcal{S}(v^*,v^*,v^*)\mathcal{S}(v^*,v^*,u^*)}{\mathcal{S}(u^*,u^*,v^*)+\mathcal{S}(v^*,v^*,u^*)} \\ &\quad + a_3\, \mathcal{S}(u^*, u^*, v^*) \\ \mathcal{S}(u^*, u^*, v^*) &\leq (a_1 + a_3)\, \mathcal{S}(u^*, u^*, v^*). \end{aligned}$$
(2.29)

Since $0 < a_1 + a_2 + a_3 < 1 \Rightarrow a_1 + a_3 < 1$, thus, we obtain $\mathcal{S}(u^*, u^*, v^*) = 0$, i.e., $u^* = v^*$.

Hence, we proved that $\mathcal{T}$ have a unique fixed point in $\mathcal{X}$.

Here completes the proof. ■

Remark 2.10.

1. If we put $a_2 = a_3 = a_4 = a_5 = 0$ in Theorem 2.6, we get the Banach Contraction Theorem [1] in S-metric space.
2. If we put $a_1 = a_2 = a_3 = a_5 = 0$ in Theorem 2.6, we get the Kanan Theorem [5] in S-metric space.
3. If we put $a_2 = a_3 = a_5 = 0$ in Theorem 2.6, we get the Fisher Theorem [24] in S-metric space.
4. If we put $a_1 = a_2 = a_3 = a_4 = 0$ in Theorem 2.6, we get the result of Chaterjee Theorem [25] in S-metric space.
5. If we put $a_2 = a_3 = 0$ in Theorem 2.9, we get the result of Dass and Gupta Theorem [26] in S-metric space.

Theorem 2.11: Let $(\mathcal{X}, \mathcal{S})$ be a complete S-metric space and $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ be a mapping such that $F(\mathcal{T}) \neq \Phi$ and that

$$\mathcal{S}(\mathcal{T}u, \mathcal{T}u, \mathcal{T}^2u) \leq \lambda\, \mathcal{S}(u, u, \mathcal{T}u)\,, \tag{2.30}$$

for all $u \in \mathcal{X}$, where $0 \leq \lambda < 1$ is a constant. Then $\mathcal{T}$ has the P property.

Proof: We always assume that $n > 1$. Since the statement for $n = 1$ is trivial. Let $w \in F(\mathcal{T}^n)$. By the hypotheses, we get

$$\begin{aligned} \mathcal{S}(w, w, \mathcal{T}w) &= \mathcal{S}(\mathcal{T}\mathcal{T}^{n-1}w, \mathcal{T}\mathcal{T}^{n-1}w, \mathcal{T}^2\mathcal{T}^{n-1}w) \\ &\leq \lambda\, \mathcal{S}(\mathcal{T}^{n-1}w, \mathcal{T}^{n-1}w, \mathcal{T}^n w) \\ &\leq \lambda\, \mathcal{S}(\mathcal{T}\mathcal{T}^{n-2}w, \mathcal{T}\mathcal{T}^{n-2}w, \mathcal{T}^2\mathcal{T}^{n-2}w) \\ &\leq \lambda^2\, \mathcal{S}(\mathcal{T}^{n-2}w, \mathcal{T}^{n-2}w, \mathcal{T}^{n-1}w) \\ &\leq \cdots \leq \lambda^n \mathcal{S}(w, w, \mathcal{T}w) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Hence $\mathcal{S}(w, w, \mathcal{T}w) = 0$, that is $\mathcal{T}w = w$.

Theorem 2.12: Under the condition of Theorem 2.3, $\mathcal{T}$ has the P property.

Proof: We have to prove that the mapping $\mathcal{T}$ satisfies (2.30). In fact, for any $u \in \mathcal{X}$, we have

$$\begin{aligned}
\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}^2u) &= \mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}\mathcal{T}u)\\
&\leq a_1\,\mathcal{S}(u,u,\mathcal{T}u) + a_2\,\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(u,u,\mathcal{T}\mathcal{T}u)+\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}\mathcal{T}u)\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}u)}{\mathcal{S}(u,u,\mathcal{T}\mathcal{T}u)+\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}u)}\\
&\quad + a_3\,\frac{\mathcal{S}(u,u,\mathcal{T}u)\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}u)+\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}\mathcal{T}u)\mathcal{S}(u,u,\mathcal{T}\mathcal{T}u)}{\mathcal{S}(u,u,\mathcal{T}\mathcal{T}u)+\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}u)}\\
&\leq a_1\,\mathcal{S}(u,u,\mathcal{T}u) + a_2\,\mathcal{S}(u,u,\mathcal{T}u) + a_3\,\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}\mathcal{T}u)\\
(1-a_3)\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}^2u) &\leq (a_1+a_2)\mathcal{S}(u,u,\mathcal{T}u)\\
\mathcal{S}(\mathcal{T}u,\mathcal{T}u,\mathcal{T}^2u) &\leq \frac{a_1+a_2}{1-a_3}\mathcal{S}(u,u,\mathcal{T}u)
\end{aligned}$$

Deduce that $\lambda = \frac{a_1+a_2}{1-a_3}$. Note that $a_1 + a_2 + a_3 < 1$, then $\lambda < 1$. Accordingly, (2.30) is satisfied. Consequently, by Theorem 2.3, $\mathcal{T}$ has the P property.

3 Conclusion

In this article, we have established the existence of fixed points theorems of rational type contractions mappings in the framework of S-metric spaces and also studied the P property for some mappings. To further demonstrated the reliability of the findings in the article, we additionally offer a few examples.

Competing Interests

Authors have declared that no competing interests exist.

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