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# On a nonlinear differential equation with two-point nonlocal condition with parameters

**A. M. A. El-Sayed[1,*], M. SH. Mohamed[1] and E. M. Al-Barg[2]**

1 Faculty of Science, Alexandria University, Alexandria, Egypt.
2 Faculty of Science, Sirt University, Libya.
* Correspondence: eman.albarg@gmail.com

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**Abstract:** Here we study the existence of solutions of a nonlocal two-point, with parameters, boundary value problem of a first order nonlinear differential equation. The maximal and minimal solutions will be proved. The continuous dependence of the unique solution on the parameters of the nonlocal condition will be proved. The anti-periodic boundary value problem will be considered as an application.

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## 1. Introduction

The existence of solutions of the two-point boundary value problem of the first order differential equation

$$\frac{dx}{dt} = f(t, x(t)), \qquad t \in (0, T)$$

have been studied in [1–3]. Also, some nonlocal problems of differential equations have been considered in [4–17]. Consider the nonlocal two-point boundary value problem with parameters $\alpha$ and $\beta$;

$$\frac{dx}{dt} = f(t, x(t)), \quad a.e, \quad t \in (0, T), \tag{1}$$

$$\alpha x(\tau) + \beta x(\eta) = x_0, \quad \tau \in [0, T), \quad \eta \in (0, T], \quad \alpha + \beta \neq 0. \tag{2}$$

Here we study the existence of at least one absolutely continuous solution $x \in AC[0, T]$ of the Problem (1)-(2). The maximal and minimal solutions of the Problem (1)-(2) will be studied. Also the continuous dependence of the unique solution $x \in AC[0, T]$ on the parameters $\alpha, \beta$ and $x_0$ will be proved. The anti-periodic boundary value problem will be considered as an application.

## 2. Existence of solutions

Consider the Problem (1)-(2) under the following assumptions;

(i) $f : [0.T] \times R \to R$ is measurable in $t \in [0, T]$ for every $x \in R$ and continuous in $x \in R$ for every $t \in [0, T]$.

(ii) There exist an integrable function $m \in L^1[0, T]$ and a constant $b \geq 0$ such that

$$|f(t, x)| \leq m(t) + b\,|x|.$$

(iii) $2bT < 1$.

### 2.1. Integral equation representation

Here we give the integral representation of the solution of the Problem (1)-(2) if it exists. We have the following lemma.

**Lemma 1.** *If the solution of the Problem (1)-(2) exists, then it can be expressed by the integral equation*

$$x(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f(s,x(s))ds - \beta \int_0^\eta f(s,x(s))ds] + \int_0^t f(s,x(s))ds. \tag{3}$$

**Proof.** Let the boundary value Problem (1)-(2). Integrating Equation (1), we obtain

$$x(t) = x(0) + \int_0^t f(s,x(s))ds. \tag{4}$$

For $t = \tau$, we obtain

$$x(\tau) = x(0) + \int_0^\tau f(s,x(s))ds$$

and

$$\alpha x(\tau) = \alpha x(0) + \alpha \int_0^\tau f(s,x(s))ds. \tag{5}$$

For $t = \eta$ in (4), we obtain

$$x(\eta) = x(0) + \int_0^\eta f(s,x(s))ds$$

and

$$\beta x(\eta) = \beta x(0) + \beta \int_0^\eta f(s,x(s))ds. \tag{6}$$

Substituting (5) and (6) into (2), we obtain

$$(\alpha+\beta)x(0) = x_0 - \alpha \int_0^\tau f(s,x(s))ds - \beta \int_0^\eta f(s,x(s))ds.$$

Then

$$x(0) = \frac{1}{\alpha+\beta}\left[x_0 - \alpha \int_0^\tau f(s,x(s))ds + \beta \int_0^\eta f(s,x(s))ds\right]$$

and

$$x(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f(s,x(s))ds - \beta \int_0^\eta f(s,x(s))ds] + \int_0^t f(s,x(s))ds. \tag{7}$$

□

Now, we have the following existence theorem

**Theorem 1.** *Let the assumption (i)-(ii)-(iii) are satisfied, then there exists at least one absolutely continuous solution $x \in AC[0,T]$ of the Problem (1)-(2).*

**Proof.** Define the operator $F$ by

$$Fx(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f(s,x(s))ds - \beta \int_0^\eta f(s,x(s))ds] + \int_0^t f(s,x(s))ds.$$

Define the set

$$Q_r = \{x : ||x|| \leq r\} \subset C[0,T], \qquad r = \frac{|x_0| + 2||m||(\alpha+\beta)}{(\alpha+\beta)(1-2b)}.$$

Let $x \in Q_r$, then

$$\begin{aligned}
|Fx(t)| &= |\frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f(s,x(s))ds - \beta \int_0^\eta f(s,x(s))ds] + \int_0^t f(s,x(s))ds| \\
&\leq \frac{1}{\alpha+\beta}[|x_0| + \alpha \int_0^\tau |f(s,x(s))|ds + \beta \int_0^\eta |f(s,x(s))|ds] + \int_0^t |f(s,x(s))|ds \\
&\leq \frac{1}{\alpha+\beta}[|x_0| + \alpha \int_0^\tau (|m(s)| + b|x|)ds + \beta \int_0^\eta (|m(s)| + b|x|)ds] + \int_0^t (|m(s)| + b|x|)ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\alpha+\beta}[|x_0|+\alpha\int_0^T(|m(s)|+b|x|)ds+\beta\int_0^T(|m(s)|+b|x|)ds]+\int_0^T(|m(s)|+b|x|)ds\\
&\leq \frac{1}{\alpha+\beta}[|x_0|+\frac{\alpha}{\alpha+\beta}(||m||+b||x||T)+\frac{\beta}{\alpha+\beta}(||m||+b||x||T)]+(||m||+b||x||T)\\
&\leq \frac{1}{\alpha+\beta}|x_0|+2(||m||+b||x||T)\leq r.
\end{aligned}$$

Then the class of functions $\{Fx\}$ is uniformly bounded on $Q_r$, and $F : Q_r \to Q_r$. Let $x \in Q_r$ and $t_1, t_2 \in [0, T]$, such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|Fx(t_2)-Fx(t_1)| &= |\frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau f(s,x(s))ds-\beta\int_0^\eta f(s,x(s))ds]+\int_0^{t_2} f(s,x(s))ds\\
&\quad -\frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau f(s,x(s))ds-\beta\int_0^\eta f(s,x(s))ds]+\int_0^{t_1} f(s,x(s))ds|\\
&\leq \int_{t_1}^{t_2}|f(s,x(s))|ds\\
&\leq \int_{t_1}^{t_2}(|m(s)|+b|x|)ds\\
&\leq \int_{t_1}^{t_2}|m(s)|ds+b|x|(t_2-t_1).
\end{aligned}$$

So, the class of functions $\{Fx\}$ is equi-continuous on $Q_r$. From Arzela Theorem [18] we deduce that the class of functions $\{Fx\}$ is compact, and $F : Q_r \to Q_r$ is compact. Now we prove that $F$ is continuous operator. For this let $\{x_n\} \subset Q_r$ be convergent sequence such that $x_n(t) \to x_0(t)$, then

$$Fx_n(t)=\frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau f(s,x_n(s))ds-\beta\int_0^\eta f(s,x_n(s))ds]+\int_0^t f(s,x_n(s))ds$$

and

$$\lim_{n\to\infty}Fx_n(t)=\frac{1}{\alpha+\beta}[x_0-\alpha\lim_{n\to\infty}\int_0^\tau f(s,x_n(s))ds-\beta\lim_{n\to\infty}\int_0^\eta f(s,x(s))ds]+\lim_{n\to\infty}\int_0^t f(s,x_n(s))ds.$$

From assumptions (i), (ii), we have

$$f(s,x_n(s))\to f(s,x_0(s))$$

and

$$|f(s,x(s))|\leq |m(s)|+b|x|\in L_1[0,T].$$

Applying Lebesgue dominated convergence Theorem [18], we have

$$\begin{aligned}
\lim_{n\to\infty}Fx_n(t) &= \frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau \lim_{n\to\infty}f(s,x_n(s))ds-\beta\int_0^\eta \lim_{n\to\infty}f(s,x_n(s))ds]+\int_0^t \lim_{n\to\infty}f(s,x_n(s))ds\\
&= \frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau f(s,\lim_{n\to\infty}x_n(s))ds-\beta\int_0^\eta f(s,\lim_{n\to\infty}x_n(s))ds]+\int_0^t f(s,\lim_{n\to\infty}x_n(s))ds\\
&= \frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau f(s,x_0(s))ds-\beta\int_0^\eta f(s,x_0(s))ds]+\int_0^t f(s,x_0(s))ds=Fx_0(t).
\end{aligned}$$

Hence, $F : Q_r \to Q_r$ is continuous. Now by Schauder fixed point Theorem [19] there exists at least one solution $x \in C[0, T]$ of the Problem (1)-(2). Let $x \in C[0, T]$ be a solution of the Problem (1)-(2). Differentiating the integral Equation (7), we obtain

$$\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt}(\frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau f(s,x(s))ds-\beta\int_0^\eta f(s,x(s))ds]+\int_0^t f(s,x(s))ds)\\
&= \frac{d}{dt}\int_0^t f(s,x(s))ds.
\end{aligned}$$

Since $f$ is measurable in $t \in [0, T]$ and bounded by integrable function, then $f \in L^1[0, T]$, and

$$\frac{dx}{dt} = f(t, x(t)) \quad a.e, \quad t \in (0, T].$$

Putting $t = \tau$ in the integral Equation (7), we get

$$x(\tau) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f(s, x(s))ds - \beta \int_0^\eta f(s, x(s))ds] + \int_0^\tau f(s, x(s))ds.$$

and

$$\begin{aligned} \alpha x(\tau) &= \frac{\alpha}{\alpha+\beta}[x_0 - \frac{\alpha_2}{\alpha+\beta} \int_0^\tau f(s, x(s))ds - \frac{\alpha\beta}{\alpha+\beta} \int_0^\eta f(s, x(s))ds] + \alpha \int_0^\tau f(s, x(s))ds \\ &= \frac{\alpha}{\alpha+\beta}x_0 + \frac{\alpha\beta}{\alpha+\beta} \int_0^\tau f(s, x(s))ds - \frac{\alpha\beta}{\alpha+\beta} \int_0^\eta f(s, x(s))ds. \end{aligned}$$

Also

$$\beta x(\eta) = \frac{\beta}{\alpha+\beta}x_0 - \frac{\alpha\beta}{\alpha+\beta} \int_0^\tau f(s, x(s))ds + \frac{\alpha\beta}{\alpha+\beta} \int_0^\eta f(s, x(s))ds.$$

Then

$$\alpha x(\tau) + \beta x(\eta) = x_0, \quad \tau \in [0, T), \eta \in (0, T].$$

□

### 2.2. Maximal and minimal solution

Let $u(t)$ be a solution of the integral Equation (7), then $u(t)$ is said to be a maximal solution of (7), if for every solution of (7) satisfies the inequality:

$$x(t) \leq u(t), \quad t \in [0, T].$$

A minimal solution $v(t)$ can be defined by similar way by reversing the above inequality *i.e.,*

$$x(t) > v(t), \quad t \in [0, T].$$

We will use the following lemma to prove the existence of the maximal and minimal solutions.

**Lemma 2.** *Let the assumption of Theorem 1 are satisfied and $x(t)$ and $y(t)$ are two continuous functions on $[0, T]$ satisfying*

$$\begin{aligned} x(t) &\leq \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f(s, x(s))ds - \beta \int_0^\eta f(s, x(s))ds] + \int_0^t f(s, x(s))ds, \\ y(t) &\geq \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f(s, y(s))ds - \beta \int_0^\eta f(s, y(s))ds] + \int_0^t f(s, y(s))ds, \end{aligned}$$

*and one of them is strict. If $f$ is monotonic nondecreasing in $x$, then*

$$x(t) < y(t), \quad t \in [0, T]. \tag{8}$$

**Proof.** Let the conclusion (8) is false, then there exist $t_1$ such that

$$x(t_1) = y(t_2) \quad t_1 > 0,$$

and

$$x(t) < y(t), \quad 0 < t < t_1.$$

From the monotonicity of $f(t, x(t))$ in $x$, we have

$$\begin{aligned} x(t_1) &\leq \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f(s, x(s))ds - \beta \int_0^\eta f(s, x(s))ds] + \int_0^{t_1} f(s, x(s))ds, \quad t \in [0, T] \\ &< \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f(s, y(s))ds - \beta \int_0^\eta f(s, y(s))ds] + \int_0^{t_1} f(s, y(s))ds, \\ &< y(t_1). \end{aligned}$$

This contradicts the fact that $x(t_1) = y(t_1)$, then

$$x(t) < y(t), \qquad t \in [0, T].$$

□

For the existence of the maximal and minimal solutions we have the following theorem.

**Theorem 2.** *Let the assumptions of Theorem 1 are satisfied. If $f(t, x(t))$ is monotonic nondecreasing in $x$ for each $t \in [0, T]$, then the Equation (7) (consequently the Problem (1)-(2)) has maximal and minimal solutions.*

**Proof.** Firstly we shall prove the existence of the maximal solution of (7). Let $\epsilon > 0$ be given then consider the integral equation

$$x_\epsilon(t) \leq \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_\epsilon(s, x_\epsilon(s))ds - \beta \int_0^\eta f_\epsilon(s, x_\epsilon(s))ds] + \int_0^{t_1} f(s, x(s))ds, \tag{9}$$

where

$$f_\epsilon(t, x_\epsilon(t)) = f(t, x_\epsilon(t)) + \epsilon.$$

It is clear that the Equation (9) has at least one solution $x_\epsilon(t) \in C[0, T]$. Now, let $\epsilon_1$ and $\epsilon_2$ be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$, then

$$\begin{aligned} x_{\epsilon_2}(t) &= \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_{\epsilon_2}(s, x_{\epsilon_2}(s))ds - \beta \int_0^\eta f_{\epsilon_2}(s, x_{\epsilon_2}(s))ds] + \int_0^t f_{\epsilon_2}(s, x_{\epsilon_2}(s))ds \\ &= \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau (f(s, x_{\epsilon_2}(s)) + \epsilon_2)ds - \beta \int_0^\eta (f(s, x_{\epsilon_2}(s)) + \epsilon_2)ds] + \int_0^t (f(s, x_{\epsilon_2}(s)) + \epsilon_2)ds. \end{aligned}$$

Also

$$\begin{aligned} x_{\epsilon_1}(t) &= \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_{\epsilon_1}(s, x_{\epsilon_1}(s))ds - \beta \int_0^\eta f_{\epsilon_1}(s, x_{\epsilon_1}(s))ds] + \int_0^t f_{\epsilon_1}(s, x_{\epsilon_1}(s))ds \\ &= \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau (f(s, x_{\epsilon_1}(s)) + \epsilon_1)ds - \beta \int_0^\eta (f(s, x_{\epsilon_1}(s)) + \epsilon_1)ds] + \int_0^t (f(s, x_{\epsilon_1}(s)) + \epsilon_1)ds \\ &> \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau (f(s, x_{\epsilon_2}(s)) + \epsilon_2)ds - \beta \int_0^\eta (f(s, x_{\epsilon_2}(s)) + \epsilon_2)ds] + \int_0^t (f(s, x_{\epsilon_2}(s)) + \epsilon_2)ds. \end{aligned}$$

Applying Lemma 2, we obtain

$$x_{\epsilon_2} < x_{\epsilon_1}, \quad t \in [0, T].$$

As shown before the family of function $x_\epsilon(t)$ is equi-continuous and uniformly bounded. Then, by Arzela Theorem [18], there exist a decreasing sequence $\epsilon_n$ such that $\epsilon_0 \to 0$ as $n \to \infty$, and $u(t) = \lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in $[0, T]$ and denote his limit by $u(t)$. From the continuity of the functions $f_\epsilon(t, x_\epsilon(t))$, we get $f_\epsilon(t, x_\epsilon(t)) \to f(t, x(t))$ as $n \to \infty$ and

$$u(t) = \lim_{n\to\infty} x_{\epsilon_n}(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f(s, x(s))ds - \beta \int_0^\eta f(s, x(s))ds] + \int_0^t f(s, x(s))ds.$$

Now we prove that $u(t)$ is the maximal solution of (7). To do this, let $x(t)$ be any solution of (7), then

$$\frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau f(s,x(s))ds-\beta\int_0^\eta f(s,x(s))ds]+\int_0^t f(s,x(s))ds \tag{10}$$

and

$$\begin{aligned} x_\epsilon(t) &= \frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau f_\epsilon(s,x_\epsilon(s))ds-\beta\int_0^\eta f_\epsilon(s,x_\epsilon(s))ds]+\int_0^t f_\epsilon(s,x_\epsilon(s))ds \\ &= \frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau (f(s,x_\epsilon(s)+\epsilon))ds-\beta\int_0^\eta (f(s,x_\epsilon(s))+\epsilon)ds]+\int_0^t (f(s,x_\epsilon(s))+\epsilon)ds \\ &> \frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau f(s,x_\epsilon(s))ds-\beta\int_0^\eta f(s,x_\epsilon(s))ds]+\int_0^t f(s,x_\epsilon(s))ds = x(t). \end{aligned}$$

Applying Lemma 2, we obtain

$$x(t) < x_\epsilon(t) \quad t \in [0,T].$$

From the uniqueness of the maximal solution, it is clear that $x_\epsilon(t)$ tends to $u(t)$ uniformly in $[0,T]$ as $\epsilon \to 0$. By similar way as done above we can prove the existence of the minimal solution. □

### 2.3. Uniqueness of the solution

Consider the problem (1)-(2) under the following assumptions

($i^*$) $f : [0,T]\times R \to R$ is measurable in $t \in [0,T]$ for every $x \in R$ satisfied the Lipschitz condition

$$|f(t,x)-f(t,y)| \le L|x-y|, \quad \text{and} \quad |f(t,0)| = m(t).$$

**Theorem 3.** *Let the assumptions ($i^*$) be satisfied. If $2LT < 1$, then the solution of the nonlocal two-point boundary value Problem (1)-(2) is unique.*

**Proof.** From assumption ($i^*$) we get

$$|f(t,x)|-|f(t,0)| \le |f(t,x)-f(t,0)| \le b|x|$$

and

$$|f(t,x)| \le b|x| + |f(t,0)| = m(t)+b|x|.$$

Then the assumptions (ii) is satisfied, so there exists at least one solution $x \in AC[0,T]$ of the Problem (1)-(2). Let $x$ and $y$ be two solutions of the Problem (1)-(2), then we have

$$\begin{aligned} |x(t)-y(t)| &= |\frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau\int_0^t f(s,x(s))ds-\beta\int_0^\eta f(s,x(s))ds]+\int_0^t f(s,x(s))ds \\ &\quad -\frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau\int_0^t f(s,y(s))ds-\beta\int_0^\eta f(s,y(s))ds]+\int_0^t f(s,y(s))ds| \\ &\le \frac{\alpha}{\alpha+\beta}\int_0^\tau |f(s,x(s))-f(s,y(s))|ds+\frac{\beta}{\alpha+\beta}\int_0^\eta |f(s,x(s))-f(s,y(s))|ds \\ &\quad +\int_0^t |f(s,x(s))-f(s,y(s))|ds \\ &\le \frac{\alpha}{\alpha+\beta}L\int_0^\tau |x(s)-y(s)|ds+\frac{\beta}{\alpha+\beta}L\int_0^\eta |x(s)-y(s)|ds+L\int_0^t |x(s)-y(s)|ds \\ &\le \frac{\alpha}{\alpha+\beta}L\int_0^T |x(s)-y(s)|ds+\frac{\beta}{\alpha+\beta}L\int_0^T |x(s)-y(s)|ds+L\int_0^T |x(s)-y(s)|ds \\ &\le \frac{\alpha}{\alpha+\beta}LT||x-y||+\frac{\beta}{\alpha+\beta}LT||x-y||+LT||x-y||. \end{aligned}$$

Then

$$||x-y|| \le 2LT||x-y||$$

and

$$||x-y||(1-2LT) \le 0 \Rightarrow ||x-y|| = 0 \Rightarrow x = y.$$

Hence, the solution of the integral Equation (7) (consequently the Problem (1)-(2)) is unique solution $x \in AC[0,T]$. □

## 3. Continuous dependence of the solution

**Definition 1.** The solution of the nonlocal two-point boundary value Problem (1)-(2) depends continuously on $x_0$, if $\forall\, \epsilon\ > 0,\quad \exists\, \delta\ > 0$, we have

$$|x - x_0^*| \le \delta \Rightarrow\ \|x - x^*\| \le \epsilon.$$

where $x^* \in AC[0,T]$ is the unique solution of the nonlocal two-point boundary value Problem (1)-(2).

**Theorem 4.** *Let the assumption of Theorem 3 are satisfied, then the solution of a nonlocal two-points boundary value Problem (1)-(2) is dependence continuously on $x_0$.*

**Proof.** let $x, x^*$ be the solutions of a nonlocal two-points boundary value Problem (1)-(2), then

$$\begin{aligned}
|x(t) - x^*(t)| &= |\frac{1}{\alpha+\beta}[x_0 - \beta\int_0^\tau f(s,x(s))ds - \beta\int_0^\eta f(s,x(s))ds] + \int_0^t f(s,x(s))ds \\
&\quad -\frac{1}{\alpha+\beta}[x_0^* - \beta\int_0^\tau f(s,x^*(s))ds - \beta\int_0^\eta f(s,x^*(s))ds] + \int_0^t f(s,x^*(s))ds| \\
&\le \frac{1}{\alpha+\beta}|x_0 - x_0^*| + \frac{\alpha}{\alpha+\beta}\int_0^\tau |f(s,x(s)) - f(s,x^*(s))|ds \\
&\quad +\frac{\beta}{\alpha+\beta}\int_0^\eta |f(s,x(s)) - f(s,x^*(s))|ds + \int_0^t |f(s,x(s)) - f(s,x^*(s))|ds \\
&\le \frac{1}{\alpha+\beta}|x_0 - x_0^*| + \frac{\alpha}{\alpha+\beta}L\int_0^T |x(s) - x^*(s)|ds \\
&\quad +\frac{\beta}{\alpha+\beta}L\int_0^T |x(s) - x^*(s)|ds + L\int_0^T |x(s) - x^*(s)|ds \\
&\le \frac{\delta}{\alpha+\beta} + 2LT||x - x^*||.
\end{aligned}$$

Then

$$||x - x^*||(1-2LT) \le \frac{\delta}{\alpha+\beta} \quad \text{implies} \quad ||x - x^*|| \le \frac{\delta}{(1-2LT)(\alpha+\beta} = \epsilon.$$

This prove the continuous dependence of solution of the nonlocal two-point boundary value Problem (1)-(2) on $x_0$. □

**Definition 2.** The solution of the nonlocal two-point boundary value Problem (1)-(2) depends continuously on $\alpha$ and $\beta$, if $\forall\, \epsilon\ > 0$, $\exists\, \delta\ > 0$, we have

$$|\alpha - \alpha^*| \le \delta_1, \quad |\beta - \beta^*| \le \delta_2 \Rightarrow\ \|x - x^*\| \le \epsilon,$$

where $x^*$ is the unique solution of the nonlocal two-points boundary value Problem (1)-(2).

**Theorem 5.** *Let the assumption of Theorem 3 is satisfied, then the solution of the nonlocal two-point boundary value Problem (1)-(2) is depends continuously on $\alpha$, $\beta$.*

**Proof.** let $x, x^*$ be the solutions of the nonlocal two-points boundary value Problem (1)-(2), then

$$\begin{aligned}
|x(t) - x^*(t)| &= |\frac{x_0}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}\int_0^\tau f(s,x(s))ds - \frac{\beta}{\alpha+\beta}\int_0^\eta f(s,x(s))ds \\
&\quad + \int_0^t f(s,x(s))ds - \frac{1}{\alpha^*+\beta^*}x_0 + \frac{\alpha^*}{\alpha^*+\beta^*}\int_0^\tau f(s,x^*(s))ds \\
&\quad + \frac{\beta^*}{\alpha^*+\beta^*}\int_0^\eta f(s,x^*(s))ds - \int_0^t f(s,x^*(s))ds| \\
&= |\frac{(\alpha^*-\alpha)(\beta^*-\beta)}{(\alpha+\beta)(\alpha^*+\beta)}x_0 - \frac{\alpha}{\alpha+\beta}\int_0^\tau f(s,x(s))ds + \frac{\alpha}{\alpha+\beta}\int_0^\tau f(s,x^*(s))ds \\
&\quad - \frac{\alpha}{\alpha+\beta}\int_0^\tau f(s,x^*(s))ds - \frac{\beta}{\alpha+\beta}\int_0^\eta f(s,x(s))ds - \frac{\beta}{\alpha+\beta}\int_0^\eta f(s,x^*(s))ds \\
&\quad - \frac{\beta}{\alpha^*+\beta}\int_0^\eta f(s,x^*(s))ds + \frac{\beta}{\alpha^*+\beta}\int_0^\tau f(s,x^*(s))ds \\
&\quad + \frac{\beta^*}{\alpha^*+\beta}\int_0^\eta f(s,x^*(s))ds + \int_0^t (f(s,x(s)) - f(s,x^*(s)))ds| \\
&\leq \frac{(\alpha^*-\alpha)(\beta^*-\beta)}{(\alpha+\beta)(\alpha^*+\beta)}|x_0| + \frac{\alpha}{\alpha+\beta}\int_0^\tau |f(s,x^*(s)) - f(s,x(s))|ds \\
&\quad + \frac{\beta}{\alpha+\beta}\int_0^\eta |f(s,x(s)) - f(s,x^*(s))|ds + \frac{\alpha^*(\alpha+\beta) - \alpha(\alpha^*+\beta^*)}{(\alpha+\beta)(\alpha^*+\beta)}\int_0^\tau |f(s,x^*(s))|ds \\
&\quad + \frac{\beta^*(\alpha+\beta) - \beta(\alpha^*+\beta^*)}{(\alpha+\beta)(\alpha^*+\beta)}\int_0^\eta |f(s,x^*(s))|ds + \int_0^t (f(s,x(s)) - f(s,x^*(s)))ds| \\
&\leq \frac{\delta_1+\delta_2}{(\alpha+\beta)(\alpha^*+\beta)}|x_0| + \frac{\alpha}{\alpha+\beta}L\int_0^T |x(s) - x^*(s)|ds \\
&\quad + \frac{\beta}{\alpha+\beta}L\int_0^T |x(s) - x^*(s)|ds + \frac{\alpha^*(\alpha+\beta) - \alpha(\alpha^*+\beta^*)}{(\alpha+\beta)(\alpha^*+\beta)}\int_0^T |f(s,x^*(s))|ds \\
&\quad + \frac{\beta^*(\alpha+\beta) - \beta(\alpha^*+\beta^*)}{(\alpha+\beta)(\alpha^*+\beta)}\int_0^T |f(s,x^*(s))|ds + L\int_0^T |x(s) - x^*(s)|ds.
\end{aligned}$$

Then

$$\begin{aligned}
||x - x^*|| &\leq \frac{(\delta_1+\delta_2}{(\alpha+\beta)(\alpha^*+\beta^*)}|x_0| + \frac{\alpha}{\alpha+\beta}LT||x-x^*|| + \frac{\beta}{\alpha+\beta}LT||x-x^*|| + \frac{\alpha^*\beta - \alpha\beta^*}{(\alpha+\beta)(\alpha^*+\beta^*)}M \\
&\quad + \frac{\beta^*\alpha - \beta\alpha^*}{(\alpha+\beta)(\alpha^*+\beta^*)}M + LT||x-x^*|| \\
&\leq \frac{\delta_1+\delta_2}{(\alpha+\beta)(\alpha^*+\beta^*)}|x_0| + 2LT||x-x^*||
\end{aligned}$$

and

$$(1-2LT)||x-x^*|| \leq \frac{\delta_1+\delta_2}{(\alpha+\beta)(\alpha^*+\beta^*)}|x_0| \quad \text{implies} \quad ||x-x^*|| \leq \frac{|\delta_1+\delta_2}{(\alpha+\beta)(\alpha^*+\beta^*)(1-2LT)}|x_0| = \epsilon.$$

This prove that $\forall \epsilon > 0, \quad \exists \delta(\epsilon) > 0$ such that

$$|\alpha - \alpha^*| \leq \delta_1, |\beta - \beta^*| \leq \delta_2 \Rightarrow ||x - x^*|| \leq \epsilon.$$

□

## 4. Anti-periodic boundary value problem

Consider the nonlocal boundary value problem of the differential equation (1) with the anti-periodic nonlocal condition

$$x(\tau) = -x(1-\tau), \qquad \tau \in [0,T],$$

we have the following corollary;

**Corollary 6.** *If $\alpha = 1$, $\beta = 1$ and $\eta = 1 - \tau$ and $x_0 = 0$ in Theorem 1, then the anti-periodic boundary value problem*

$$\begin{aligned} \frac{dx}{dt} &= f(t, x(t)), \quad t \in (0, T) \\ x(\tau) &= -x(1-\tau), \quad \tau \in (0, T) \end{aligned}$$

*has the at lease one solution $x \in AC[0, T]$*

$$x(t) = -\int_0^\tau f(s, x(s))ds - \int_0^{1-\tau} f(s, x(s))ds + \int_0^t f(s, x(s))ds.$$

*Now, let $\tau = \frac{1}{2}$, then*

$$\begin{aligned} x(t) &= -\int_0^{\frac{1}{2}} f(s, x(s))ds - \int_0^{\frac{1}{2}} f(s, x(s))ds + \int_0^t f(s, x(s))ds \\ &= \int_0^t f(s, x(s))ds - 2\int_0^{\frac{1}{2}} f(s, x(s))ds. \end{aligned}$$

## 5. Examples

**Example 1.** Consider the differential Equation (1) with the backward condition $x(T) = x_0$, we have the following corollary;

**Corollary 7.** *Let $\alpha = 1$, $\beta = 0$ and $\eta = T$ in Theorem 1, then the backward problem*

$$\frac{dx}{dt} = f(t, x(t))$$

$$x(T) = x_0$$

*has the solution $x \in AC[0, T]$*

$$\begin{aligned} x(t) &= x_0 - \int_0^T f(s, x(s))ds + \int_0^t f(s, x(s))ds \\ &= x_0 - \int_t^T f(s, x(s))ds. \end{aligned}$$

**Example 2.** Consider the differential Equation (1) with the forward condition $x(0) = x_0$, we have the following corollary;

**Corollary 8.** *Let $\alpha = 0$, $\beta = 1$ and $\tau = 0$ in Theorem 1, then the initial value problem*

$$\frac{dx}{dt} = f(t, x(t))$$

$$x(0) = x_0$$

*has the solution $x \in AC[0, T]$*

$$x(t) = x_0 + \int_0^t f(s, x(s))ds$$

## 6. Conclusions

We proved here, under certain conditions, the existence of at least one absolutely continuous solution $x \in AC[0, T]$ of the nonlocal two-point, with parameters ( $\alpha, \beta$ and $x_o$ ) boundary value Problem (1)-(2). The maximal and minimal solutions of the Problem (1)-(2) have been proved. The continuous dependence of the unique solution on the parameters $\alpha, \beta$ and $x_o$ ) have been also proved. The anti-periodic boundary value problem have been considered as an application.

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