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BOUNDEDNESS OF LITTLEWOOD–PALEY OPERATORS WITH VARIABLE KERNEL ON WEIGHTED HERZ SPACES WITH VARIABLE EXPONENT

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Abstract. Let $\Omega \in L^{\infty}(\mathbb{R}^n) \times L^2(S^{n-1})$ be a homogeneous function of degree zero. In this article, we obtain some boundedness of the parameterized Littlewood−Paley operators with variable kernels on weighted Herz spaces with variable exponent.

Index Terms: Parameterized Littlewood-Paley operators; variable kernel; weighted Herz spaces; Muckenhoupt; variable exponents.

1. Introduction

The boundedness of Littlewood-Paley operators on function spaces are one of the very important tools, not only in harmonic analysis, but also in potential theory and in partial differential equations (see[1, 2, 3, 4, 5, 6], for details). In 2004, Ding, Lin and Shao [7] investigated the L^2-boundedness for a class of Marcinkiewicz integral operators with variable kernels μ_{Ω} and $\mu_{\Omega,s}$ related to the Littlewood-Paley function $\mu^*_{\Omega,\lambda}$ and the area integral g^*_{λ}. In 2006, the authors [8] proved the L^p-boundedness of the Littlewood-Paley operators with variable kernels. In 2009, Xue and Ding [9] established the weighted estimate for Littlewood-Paley operators and their commutators.

In 1960, Hörmander [10] introduced the parameterized Littlewood-Paley operators for the first time. Now, let us recall the definitions of the parameterized Lusin area integral and Littlewood-Paley g^*_{λ} function.

Let $S^{n-1}(n \geq 2)$ be the unit sphere in $\mathbb{R}^n$ with normalized Lebesgue measure $d\sigma(x')$. Take $\Omega(x,z) \in L^{\infty}(\mathbb{R}^n) \times L^r(S^{n-1})(r \geq 1)$ to be a homogeneous function

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of degree zero and

$$\int_{s^{n-1}} \Omega(x, z')\mathrm{d}\sigma(z') = 0, \text{ for all } x \in \mathbb{R}^n, \tag{1}$$

where Ω satisfies the following conditions:

(1) For any $x, z \in \mathbb{R}^n$ and any $\lambda > 0$, we have $\Omega(x, \lambda z) = \Omega(x, z)$;

(2) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})} := \sup\limits_{r\geq 0, y\in\mathbb{R}^n} \left(\int_{s^{n-1}} |\Omega(rz' + y, z')|^r \mathrm{d}\sigma(z')\right)^{\frac{1}{r}} < \infty.$

The parameterized Littlewood-Paley operators $\mu^{\rho}_{\Omega,s}$ and $\mu^{*;\rho}_{\Omega,\lambda}$ with variable kernels, which are related to the Lusin area integral and the Littlewood-Paley g^*_λ function are defined by

$$\mu^{\rho}_{\Omega,s}(f)(x) = \left(\int\int_{\Gamma(x)} \left|\frac{1}{t^\rho}\int_{|y-z|\leq t} \frac{\Omega(y, y-z)}{|y-z|^{n-\rho}} f(z)\mathrm{d}z\right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}$$

and

$$\mu^{*,\rho}_{\Omega,\lambda}(f)(x) =$$

$$\left(\int\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\frac{1}{t^\rho}\int_{|y-z|\leq t} \frac{\Omega(y, y-z)}{|y-z|^{n-\rho}} f(z)\mathrm{d}z\right|^2 \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}},$$

where $\Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |x-y| < t \text{ and } \lambda > 1\}$.

In 2013, Wei and Tao [11] investigated the boundedness of parameterized Littlewood Paley operators on weighted weak Hardy spaces. Lin and Xuan [12] established the boundedness for commutators of parameterized Littlewood-Paley operators and area integrals on weighted Lebesgue spaces $L^p(w)$.

The theory of the variable exponent function spaces has been rapidly developed after the work [13], where Kováčik and Rákosník have clarified fundamental properties of Lebesgue spaces with variable exponent. After that, many researchers have been interested in the theory of the variable exponent spaces (see [14, 15, 16, 17, 18, 19, 20]).

The generalization of the Muckenhoupt weights with variable exponent $A_{p(\cdot)}$ has been considered in [21, 22, 23, 24]. The equivalence between the Muckenhoupt condition and the boundedness of the Hardy-Littlewood maximal operator on weighted Lebesgue spaces with variable exponent were discussed in [21, 22]. After that, Cruz-Uribe and Wang [25] proved the boundedness of some classical operators on weighted Lebesgue spaces with variable exponent $L^{p(\cdot)}(w)$.

Recently, Izuki and Noi [26] introduced the weighted Herz spaces with variable exponent, and also studied the boundedness of fractional integrals on those spaces.

In this paper, we establish the boundedness of parameterized Littlewood-Paley operators with variable kernels on weighted Herz spaces with variable exponent

$\dot{K}^{\alpha,q}_{p(\cdot)}(w)$. Let E be a Lebesgue measurable set in $\mathbb{R}^n$ with measure $|E| > 0$, χ_E means its characteristic function. We shall recall some definitions.

Definition 1.1. Let $p(\cdot) : E \to [1, \infty)$ be a measurable function. The variable exponent Lebesgue space is defined as

$$L^{p(\cdot)}(E) = \{f \text{ is measurable} : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \eta > 0\}.$$

The space $L^{p(\cdot)}_{\text{loc}}(E)$ is defined as

$$L^{p(\cdot)}_{\text{loc}}(E) = \{f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset E\}.$$

The Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach spaces with the norm defined as

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \eta > 0 : \int_E \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \mathrm{d}x \leq 1 \right\}.$$

We denote $p_- = \text{ess inf}\{p(x) : x \in E\}$, $p_+ = \text{ ess sup}\{p(x) : x \in E\}$, then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Definition 1.2. [27] Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$. A measurable function $p(\cdot)$ is said to be globally log-Höder continuous if it satisfies

(1) $|p(x) - p(y)| \leq \frac{1}{-\log(|x-y|)}$, $x, y \in \mathbb{R}^n, |x - y| \leq 1/2$;

(2) $|p(x) - p_\infty| \leq \frac{1}{\log(e+|x|)}$, $x \in \mathbb{R}^n$,

for some $p_\infty \geq 1$. The set of $p(\cdot)$ satisfying conditions (1) and (2) is denoted by $LH(\mathbb{R}^n)$.

We know that, if $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (see[28]).

Definition 1.3. [29] Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w is a weight function. The weighted Lebesgue spaces with variable exponent $L^{p(\cdot)}(w)$ is the set of all complex-valued measurable function f such that $fw^{1/p(\cdot)} \in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(w)$ is a Banach space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(w)} = \|fw^{1/p(\cdot)}\|_{L^{p(\cdot)}}.$$

$p'(\cdot)$ is the conjugate of $p(\cdot)$ such that $\frac{1}{p'(\cdot)} + \frac{1}{p(\cdot)} = 1$. Next, we introduce the classical Muckenhoupt A_p weight.

Definition 1.4. [30] Let $1 < p < \infty$, then $w \in A_p$ for every cube Q,

$$\left(\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\mathrm{d}x\right)^{p-1} \leq C < \infty.$$

We say that $w \in A_1$ if it satisfies $Mw(x) \leq w(x)$ for all $x \in \mathbb{R}^n$. The set A_1 consists of all Muckenhoupt A_1 weights.

Definition 1.5. [21, 25] Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and a weight w, then $w \in A_{p(\cdot)}$ if

$$\sup_{\text{B:ball}} |B|^{-1} \|w^{1/p(\cdot)}\chi_B\|_{L^{p(\cdot)}(w)} \|w^{-1/p(\cdot)}\chi_B\|_{L^{p'(\cdot)}(w)} < \infty.$$

Definition 1.6. [25] Given $p_1(\cdot),\ p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $1/p_1(x) - 1/p_2(x) = \mu/n$ such that $0 < \mu < n$. Then $w \in A_{p_1(\cdot),p_2(\cdot)}$ if

$$\|w\chi_B\|_{L^{p_2(\cdot)}} \|w^{-1}\chi_B\|_{L^{p_1'(\cdot)}} \leq |B|^{\frac{n-\mu}{n}}$$

holds for all balls $B \in \mathbb{R}^n$.

Definition 1.7. [25] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight. We say that $(p(\cdot), w)$ is an M-pair if the maximal operator M is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-1})$.

Now, we to need to give the definition of weighted Herz space with variable exponent. For all $k \in \mathbb{Z}$, we denote $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \backslash B_{k-1}$, $\chi_k = \chi_{C_k}$.

Definition 1.8. [26] Suppose that $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $0 < q < \infty$, $\alpha \in \mathbb{R}$. The homogeneous weighted Herz space with variable exponent $\dot{K}^{\alpha,q}_{p(\cdot)}(w)$ is the collection of $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n\backslash\{0\}, w)$ such that

$$\|f\|_{\dot{K}^{\alpha,q}_{p(\cdot)}(w)} := \left(\sum_{k=-\infty}^{\infty} 2^{\alpha q k} \|f\chi_k\|^q_{L^{p(\cdot)}(w)}\right)^{1/q} < \infty.$$

It is easy to see that if $w = 1$, then $\dot{K}^{\alpha,q}_{p(\cdot)}(w) = \dot{K}^{\alpha,q}_{p(\cdot)}(\mathbb{R}^n)$ is the Herz space with variable exponen [17]. If $w = 1$ and $p(\cdot) = p$, then $\dot{K}^{\alpha,q}_{p(\cdot)}(w) = \dot{K}^{\alpha,q}_{p}(\mathbb{R}^n)$ is the classical Herz space introduced in [31]. If $p(\cdot) = p$, then $\dot{K}^{\alpha,q}_{p(\cdot)}(w) = \dot{K}^{\alpha,q}_{p}(w)$ is the weighted Herz space [32].

Definition 1.9. We say a kernel function $\Omega(x, z)$ satisfies the L^r-Dini condition $(r \geq 1)$, if

$$\int_0^1 \frac{\omega_r(\delta)}{\delta}(1 + |\log\delta|^\sigma)\mathrm{d}\delta < \infty,$$

where $\omega_r(\delta)$ denotes the integral modulus of continuity of order r of Ω defined by

$$\omega_r(\delta) = \sup_{x\in\mathbb{R}^n, |\rho|<\delta} \left(\int_{S^{n-1}} |\Omega(x, \rho z') - \Omega(x, z')|^r \mathrm{d}\sigma(z')\right)^{\frac{1}{r}},$$

where ρ is the rotation in $\mathbb{R}^n$, $\|\rho\| = \sup\limits_{z'\in S^{n-1}} \|\rho z' - z'\|$.

2. Preliminaries and notations

In order to prove our main theorems, we need the following Lemmas.

Lemma 2.1. [3] *Suppose that $X \subset \mathcal{M}$ is a Banach function space.*

(1) *(The generalized Hölder inequality) For all $f \in X$ and $g \in X'$, we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \leq \|f\|_X \|g\|_X.$$

(2) *For all $f \in X$, we have*

$$\sup \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x \right| : \|g\|_{X'} \leq 1 \right\} = \|f\|_X.$$

In particular, space $(X')' = X$.

As an application of the generalized Hölder inequality above, we have the following Lemma.

Lemma 2.2. *Let X be a Banach function space, we have*

$$1 \leq \frac{1}{|B|}\|\chi_B\|_X\|\chi_B\|_{X'}$$

hold for all balls B.

Lemma 2.3. [24] *Let X be a Banach function space. If the Hardy-Littlewood maximal operator M is weakly bounded on X, that is*

$$\|\chi_{\{Mf>\lambda\}}\|_X \leq \lambda^{-1}\|f\|_X,$$

holds for all $f \in X$ and $\lambda > 0l$, then we get

$$\sup_{\text{B:Ball}} \frac{1}{|B|}\|\chi_B\|_X\|\chi_B\|_{X'} < \infty.$$

Remark 2.1. [29] The weighted Banach function space $X(\mathbb{R}^n, W)$ is a Banach function space equipped the norm $\|f\|_{X(\mathbb{R}^n,W)} := \|fW\|_X$. The associated space of $X(\mathbb{R}^n, W)$ is a Banach function space and equals $X'(\mathbb{R}^n, W^{-1})$.

Remark 2.2. If $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, by comparing the definition of the weighted Banach function space with weighted variable Lebesgue space, we have

(1) If $X = L^{p_1(\cdot)}(\mathbb{R}^n)$ and $W = w$, then we obtain

$$L^{p_1(\cdot)}(\mathbb{R}^n, w) = L^{p_1(\cdot)}(w^{p_1(\cdot)}).$$

(2) If $X = L^{p_1'(\cdot)}(\mathbb{R}^n)$ and $W = w^{-1}$. Using Lemma 2.3, we obtain

$$L^{p_1'(\cdot)}(\mathbb{R}^n, w^{-1}) = L^{p_1'(\cdot)}(w^{-p_1'(\cdot)}) = (L^{p_1(\cdot)}(w^{p_1(\cdot)}))'.$$

Lemma 2.4. [33] *Suppose that X is a Banach space. Let M be bounded on the associated space X'. Then there exists a constant $0<\delta<1$ such that*

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \leq \left(\frac{|E|}{|B|}\right)^{\delta}$$

holds for all balls B and all measurable sets $E \subset B$.

Lemma 2.5. [26] *Suppose that $p_1(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w^{p_1(\cdot)} \in A_1$. Let M be a bounded on $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ and $L^{p_1'(\cdot)}(w^{-p_1'(\cdot)})$, then there exist constants $\delta_1, \delta_2 \in (0,1)$ such that*

$$\frac{\|\chi_S\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}}{\|\chi_B\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_1}, \frac{\|\chi_S\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_B\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}} \leq C\left(\frac{|S|}{|B|}\right)^{\delta_2},$$

hold for all balls B and all measurable sets $S \subset B$.

Lemma 2.6. [34] *Suppose that $\Omega \in L^{\infty}(\mathbb{R}^n) \times L^2(S^{n-1})$ satisfies equation* (1) *and definition (1.9), $\lambda > 2$, $2\rho - n > 0$, $1 < p < \infty$. Then for all $f \in L^q(w)$ there exists $C > 0$ independent of f such that*

$$\|\mu^{\rho}_{\Omega,s} f\|_{L^q(w)} \leq C\|f\|_{L^q(w)}$$

and

$$\|\mu^{*,\rho}_{\Omega,\lambda} f\|_{L^q(w)} \leq C\|f\|_{L^q(w)}.$$

Lemma 2.7. [25] *Assume that for p_0, $1 < p_0 < \infty$ and every $w_0 \in A_{p_0}$,*

$$\int\limits_{\mathbb{R}^n} f(x)^{p_0} w_0(x)\mathrm{d}x \leq \int\limits_{\mathbb{R}^n} g(x)^{p_0} w_0(x)\mathrm{d}x, \qquad (f,g) \in \mathcal{F}.$$

Then for any M-pair $(p(\cdot), w)$,

$$\|f\|_{L^{p(\cdot)}(w)} \leq C\|g\|_{L^{p(\cdot)}(w)}, \qquad (f,g) \in \mathcal{F},$$

Lemma 2.7 holds for $p_0 = 1$ and the maximal operator is bounded on $L^{p'(\cdot)}(w^{-1})$. We know the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$ (see [33]).
Combining Lemma 2.6 with Lemma 2.7, we obtain the following conclusion.

Corollary 2.8. *Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $\Omega \in L^{\infty}(\mathbb{R}^n) \times L^r(S^{n-1}) (r \geq 1)$ and $w \in A_{p(\cdot)}$. Then the parameterized Littlewood-Paley operators $\mu^{\rho}_{\Omega,s}$ and $\mu^{*,\rho}_{\Omega,\lambda}$ with variable kernels are bounded on $L^{p(\cdot)}(w)$.*

3. Main Theorems and their proofs

In this section, we will prove the boundedness of the parameterized Littlewood-Paley operators with variable kernels on variable weighted Herz spaces.

Theorem 3.1. *Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $0 < q_1 \leq q_2 < \infty$, $\lambda > 2$, $2\rho - n > 0$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ satisfies* (1.1) *and* (1.2). *If $w^{p(\cdot)} \in A_1$ and $-n\delta_1 < \alpha < n\delta_2$, where δ_1, δ_2 are the constants in Lemma 2.5, then the operator $\mu_{\Omega,s}$ is bounded from $\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})$ to $\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})$.*

Theorem 3.2. *Let $p(\cdot) \in LH(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$, $0 < q_1 \leq q_2 < \infty$, $\lambda > 2$, $2\rho - n > 0$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^2(S^{n-1})$ satisfies* (1.1) *and* (1.2). *If $w^{p(\cdot)} \in A_1$ and $-n\delta_1 < \alpha < n\delta_2$, where δ_1, δ_2 are the constants in Lemma 2.5, then the operator $\mu^*_{\Omega,\lambda}$ is bounded from $\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})$ to $\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})$.*

Remark 3.1. As it is well known that, $\mu^\rho_{\Omega,s} f(x) \leq 2^{n\lambda} \mu^{*,\rho}_{\Omega,\lambda} f(x)$ (see [6], p.89). Therefore, we give only the proof of Theorem 3.2.

Proof of Theorem 3.2

Let $f \in \dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})$. By the Jensen inequality, we have

$$\begin{aligned}
\|\mu^{*,\rho}_{\Omega,\lambda} f\|^{q_1}_{\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})} &\leq \left(\sum_{k=-\infty}^{\infty} 2^{\alpha q_2 k} \|(\mu^{*,\rho}_{\Omega,\lambda} f)\chi_k\|^{q_2}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{\frac{q_1}{q_2}} \\
&\leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \|(\mu^{*,\rho}_{\Omega,\lambda} f)\chi_k\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}$$

Denote $f_j = f\chi_j$ for each $j \in z$, then $f = \sum_{j=-\infty}^{\infty} f_j$, so we have

$$\begin{aligned}
\|\mu^{*,\rho}_{\Omega,\lambda} f\|^{q_1}_{\dot{K}^{\alpha,q_2}_{p_1(\cdot)}(w^{p_1(\cdot)})} &\leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \|(\mu^{*,\rho}_{\Omega,\lambda} f)\chi_k\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=-\infty}^{k-2} \|(\mu^{*,\rho}_{\Omega,\lambda} f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=k-2}^{k+2} \|(\mu^{*,\rho}_{\Omega,\lambda} f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\leq \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=k+2}^{\infty} \|(\mu^{*,\rho}_{\Omega,\lambda} f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&=: \mathrm{L}_1 + \mathrm{L}_2 + \mathrm{L}_3.
\end{aligned}$$

First, we consider L_2. Using Lemma 2.1 and $-2 \leq k - j \leq 2$, it is easy to get

$$\begin{aligned}
\mathrm{L}_2 &= \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=k-2}^{k+2} \|(\mu^{*,\rho}_{\Omega,\lambda} f_j)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&= \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-2}^{k+2} 2^{\alpha(k-j)} 2^{\alpha j} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1}
\end{aligned}$$

$$\leq C\|f\|^{q_1}_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.$$

Now we need to consider $\mu^{*,\rho}_{\Omega,\lambda}f_j$. Applying the Minkowski inequality, we conclude that

$$\begin{aligned}
&|\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x)| = \\
&\left(\int_0^\infty\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\rho}\int_{|y-z|\leq t}\frac{\Omega(y,y-z)}{|y-z|^{n-\rho}}f_j(z)\mathrm{d}z\right|^2\frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}}\right)^{\frac{1}{2}}\\
&\quad\leq\int_{\mathbb{R}^n}f_j(z)\left(\int_0^\infty\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho+n+1}}\right)^{\frac{1}{2}}\mathrm{d}z\\
&\quad\leq\int_{\mathbb{R}^n}f_j(z)\left(\int_0^{|x-z|}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho+n+1}}\right)^{\frac{1}{2}}\mathrm{d}z\\
&\quad+\int_{\mathbb{R}^n}f_j(z)\left(\int_{|x-z|}^\infty\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho+n+1}}\right)^{\frac{1}{2}}\mathrm{d}z.
\end{aligned}$$

For $\Omega\in L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})$ and $2\rho-n>0$, the following inequality holds

$$\begin{aligned}
\int_{|y-z|\leq t}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\mathrm{d}y &\leq \int_{s^{n-1}}\int_0^t\frac{|\Omega(sy'+z,y')|^2}{s^{2n-2\rho}}s^{n-1}\mathrm{d}s\mathrm{d}\sigma(y')\\
&\leq\|\Omega\|^2_{L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})}t^{2\rho-n}.
\end{aligned}$$

Since $|x-z|\leq|x-y|+|y-z|\leq|x-y|+t$. For $\lambda>2$, taking $0<\delta<(\lambda-2)n$, we have

$$\begin{aligned}
&\int_0^{|x-z|}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho+n+1}}\\
&\leq\int_0^{|x-z|}\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n-2n-\delta}\frac{1}{|x-z|^{2n+\delta}}\\
&\times\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho-n-\delta+1}}\\
&\leq\frac{1}{|x-z|^{2n+\delta}}\int_0^{|x-z|}\int_{|y-z|\leq t}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho-n-\delta+1}}\\
&\leq\frac{\|\Omega\|^2_{L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})}}{|x-z|^{2n+\delta}}\int_0^{|x-z|}t^{\delta-1}\mathrm{d}t\\
&\leq C|x-z|^{-2n}. \qquad (1.3)
\end{aligned}$$

If we take $1<\lambda_1<2$, then $\lambda_1 n-n>0$ and $\lambda_1 n-2n<0$, so we have

$$\begin{aligned}
&\int_{|x-z|}^\infty\int_{|y-z|\leq t}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho+n+1}}\\
&\leq\int_{|x-z|}^\infty\int_{|y-z|\leq t}|x-z|^{-\lambda_1 n}\frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-2\rho}}\frac{\mathrm{d}y\mathrm{d}t}{t^{2\rho-\lambda_1 n+n+1}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{|x-z|}^{\infty} |x-z|^{-\lambda_1 n} \int_{|y-z|\leq t} \frac{|\Omega(y,y-z)|^2}{|y-z|^{2n-\lambda_1 n}} \frac{\mathrm{d}y\mathrm{d}t}{t^{n+1}} \\
&\leq \int_{|x-z|}^{\infty} |x-z|^{-\lambda_1 n} \int_{s^{n-1}} \int_0^t \frac{|\Omega(y',(y-z)')|^2}{s^{2n-\lambda_1 n}} s^{n-1} \mathrm{d}s\mathrm{d}\sigma(y') \frac{\mathrm{d}t}{t^{n+1}} \\
&\leq C\|\Omega\|^2_{L^\infty(\mathbb{R}^n)\times L^2(S^{n-1})} |x-z|^{-\lambda_1 n} \int_{|x-z|}^{\infty} t^{\lambda_1 n-2n-1}\mathrm{d}t \\
&\leq C|x-z|^{-2n}. \qquad (1.4)
\end{aligned}
$$

Combining the above two estimates, we obtain

$$
\mu^*_{\Omega,\lambda}(f)(x) \leq \int_{\mathbb{R}^n} \frac{|f(z)|}{|x-z|^n}\mathrm{d}z. \qquad (1.5)
$$

Next, we consider L_1. Noting that for $x \in A_k$, $z \in A_j$ and $j \leq k-2$, then $|x-z| \sim |x|$. By the virtue of the generalized Hölder inequality, we have

$$
\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x) \leq C2^{-kn}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}.
$$

Applying Lemma 2.3 and Lemma 2.5, we take $\|.\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}$ for each side, we have

$$
\begin{aligned}
&\|\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\leq C2^{-kn}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\leq C2^{-kn}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\leq C\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \\
&\leq C\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}}{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}} \\
&\leq C2^{(j-k)n\delta_2}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mathrm{L}_1 &\leq C\sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k}\left(\sum_{j=-\infty}^{k-2} 2^{(j-k)n\delta_2}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1} \\
&\leq C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha}2^{(k-j)(\alpha-n\delta_2)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1}.
\end{aligned}
$$

Now we have two cases: $1 < q_1 < \infty$ and $0 < q_1 \leq 1$. When $1 < q_1 < \infty$, by using the Hölder inequality, we have

$$
\mathrm{L}_1 \leq C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2} 2^{j\alpha}2^{(k-j)(\alpha-n\delta_2)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1}
$$

$$
\begin{aligned}
&\leq C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha q_1}2^{(k-j)(\alpha-n\delta_2)q_1/2}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)\\
&\times\left(\sum_{j=-\infty}^{k-2}2^{(k-j)(\alpha-n\delta_2)q_1'/2}\right)^{q_1/q_1'}\\
&\leq C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{j\alpha q_1}2^{(k-j)(\alpha-n\delta_2)q_1/2}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C\sum_{j=-\infty}^{\infty}2^{j\alpha q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\sum_{k=j+2}^{\infty}2^{(k-j)(\alpha-n\delta_2)q_1/2}\\
&\leq C\|f\|^{q_1}_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
$$

When $0<q_1\leq 1$, again by the Jensen inequality, we obtain

$$
\begin{aligned}
\mathrm{L}_1&\leq C\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{k-2}2^{j\alpha}2^{(k-j)(\alpha-n\delta_2)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{q_1}\\
&\leq C\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}2^{j\alpha q_1}2^{(k-j)(\alpha-n\delta_2)q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C\sum_{j=-\infty}^{\infty}2^{j\alpha q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\sum_{k=j+2}^{\infty}2^{(k-j)(\alpha-n\delta_2)q_1}\\
&\leq C\|f\|^{q_1}_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
$$

Finally, we estimate L_3. Noting that for $x\in A_k$, $y\in A_j$ and $j\geq k+2$, then $|y-x|\sim|y|$. By the virtue of the generalized Hölder inequality, we have

$$\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x)\leq C2^{-jn}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}.$$

Applying Lemma 2.3 and Lemma 2.5, we can take $\|.\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}$ for each side, we have

$$
\begin{aligned}
&\|\mu^{*,\rho}_{\Omega,\lambda}(f_j)(x)\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C2^{-jn}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_k\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C2^{-jn}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}\|\chi_{B_k}\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\\
&\leq C\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))}\|\chi_{B_j}\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))}\\
&\leq C\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\frac{\|\chi_{B_k}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))}}{\|\chi_{B_j}\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))}}\\
&\leq C2^{(k-j)n\delta_1}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mathrm{L}_3 &\leq C \sum_{k=-\infty}^{\infty} 2^{\alpha q_1 k} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_1} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha} 2^{(k-j)(\alpha+n\delta_1)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1}.
\end{aligned}
$$

Now we also have two cases: $1 < q_1 < \infty$ and $0 < q_1 \leq 1$. When $1 < q_1 < \infty$, by using the Hölder inequality, we have

$$
\begin{aligned}
\mathrm{L}_3 &\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha} 2^{(k-j)(\alpha+n\delta_1)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha q_1} 2^{(k-j)(\alpha+n\delta_1)q_1/2} \|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right) \\
&\times \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha+n\delta_1)q_1'/2} \right)^{q_1/q_1'} \\
&\leq C \sum_{k=-\infty}^{\infty} \sum_{j=k+2}^{\infty} 2^{j\alpha q_1} 2^{(k-j)(\alpha+n\delta_1)q_1/2} \|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\leq C \sum_{j=-\infty}^{\infty} 2^{j\alpha q_1} \|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{j-2} 2^{(k-j)(\alpha+n\delta_1)q_1/2} \\
&\leq C \|f\|^{q_1}_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
$$

When $0 < q_1 \leq 1$, applying the Jensen inequality, we obtain

$$
\begin{aligned}
\mathrm{L}_3 &\leq C \sum_{k=-\infty}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{j\alpha} 2^{(k-j)(\alpha+n\delta_1)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \right)^{q_1} \\
&\leq C \sum_{k=-\infty}^{\infty} \sum_{j=k+2}^{\infty} 2^{j\alpha q_1} 2^{(k-j)(\alpha+n\delta_1)q_1} \|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\
&\leq C \sum_{j=-\infty}^{\infty} 2^{j\alpha q_1} \|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \sum_{k=-\infty}^{j-2} 2^{(k-j)(\alpha-n\delta_2)q_1} \\
&\leq C \|f\|^{q_1}_{\dot{K}^{\alpha,q_1}_{p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
$$

This completes the proof of Theorem 3.2.

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