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Ideally Statistical Convergence in n-normed Space

Nazneen Khan[1*] and Amani Shatarah[1]

[1] *Department of Mathematics, College of Science and Arts, Taibah University, Madina, Saudi Arabia.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The aim of the article is to extend the concept of Ideally statistical convergence from 2 normed spaces to n-normed space. We have also study and prove some important algebraic and topological properties of $\mathcal{I}$*deally*-statistical convergence of real sequences in n-normed space. In the last part of this article we obtain a criterion for $\mathcal{I}$-statistically Cauchy sequence in n-normed space to be $\mathcal{I}$-statistically Cauchy with respect to $\|.\|_\infty$.

Keywords: Statistical convergence; Ideal convergence; Filter; statistically Cauchy sequence; real sequences; n-normed space.

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1 Introduction

The concept of statistical convergence was introduced by Steinhaus in 1951 but the extension of convergence of real sequences to statistical convergence was given by Fast [1]. We can find its

*Corresponding author: E-mail: nazneen4maths@gmail.com;

applications in many areas of mathematics like number theory, trigonometric series and summability theory. Also Maddox [2] extended the concept for sequences in any Hausdorff locally convex topological vector spaces. In the case of real sequences, Fridy and Orhan [3] obtained the statistical analogue of the Cauchy criterion for convergence.

Let K be a subset of $\mathbb{N}$. Then the asymptotic density of K is denoted by

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n}|k \leq n : k \in K|$$

where the vertical bars indicate the cardinality of the set.

A sequence $x = (x_k)$ is called *Statistically Convergent* to L if for $\epsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}) = 0.$$

We write this as $st - \lim_{k\to\infty} x_k = L$

It is known that, $\mathcal{I}$-convergence (where $\mathcal{I}$ stands for ideal) is generalization of the statistical convergence and it was introduced by Kostyrko et al. in [4]. It was further studied by Demirci[5], Das et al. [6], Šalát et al. [7], and many others.

Definition 1.1. *A family of sets $\mathcal{I} \subseteq 2^X$ (power set of X) is said to be an ideal in X if*
1) $\phi \in \mathcal{I}$.
2) $\mathcal{I}$ is additive i.e $\forall A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
3) $\mathcal{I}$ is hereditary i.e $\forall A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$.

A non-trivial ideal $\mathcal{I}$ is called admissible if $\{\{x\} : x \in X\} \subseteq \mathcal{I}$. $\mathcal{I}$ is maximal if there cannot exist any non-trivial ideal $J \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset.

Definition 1.2. *A non-empty family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is said to be filter on $\mathbb{N}$ if and only if*
1) $\phi \notin \mathcal{F}$,
2) $\forall\ A,\ B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$,
3) $\forall\ A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$.

If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N}\backslash A\}$ is a filter of $\mathbb{N}$. It is called as filter associated with the ideal.

Let $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural, real and complex numbers respectively. The set of all sequences is denoted by ω. Any subset of the ω is called sequence space. A sequence$(\mathrm{x}_k) \in \omega$ is said to be $\mathcal{I}$-convergent to a number L if for every $\epsilon > 0$, $\{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in \mathcal{I}$. In this case we write $\mathcal{I} - \lim x_k =$ L.

For more details please see [8] and [9]

The theory of 2-normed spaces was first introduced by Gähler[10] in 1964. Later on it was extended to n-normed spaces by Misiak [11] . Since then many mathematicians have worked in this field and obtained many interesting results for instance see Gunawan[12],[13], Gunawan and Mashadi[14], Yamanci and Gürdal [15], E. Savas([16], [17], [18]) and so on. Let $n \in \mathbb{N}$ and X^n be a linear metric space over the field $\mathbb{K}$ of real or complex numbers of dimension d, where $d \geq n \geq 2$.

Definition 1.3. *A real valued function $\|\cdot, ..., \cdot\|$ on X^n satisfying the following four conditions:*
(1)$\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent;

(2) $\|x_1, x_2, ..., x_n\|$ *is invariant under permutation;*
(3) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$ *for any* $\alpha \in \mathbb{K}$*; and*
(4) $\|x + x', x_2, ..., x_n\| \leq \|x, x_2, ..., x_n\| + \|x', x_2, ..., x_n\|$.

is called an $n-$*norm on* X *and the pair* $(X, \|\cdot, ..., \cdot\|)$ *is called an* $n-$*normed space over the field* $\mathbb{K}$.

Example 1.4. *If we take* $X = \mathbb{R}^n$*, equipped with Euclidean n-norm*

$$\|x_1, x_2, ..., x_n\|_E = vol(n - dimensional\ parallelepiped)$$

spanned by the vectors $x_1, x_2, ..., x_n$.
It may be given by the formula $\|x_1, x_2, ..., x_n\|_E = |det(x_{ij})|$*, where* $x_{ij} = (x_{i1}, x_{i2}, ..., x_{in}) \in \mathbb{R}^n$ *for each* $i = 1, 2, 3, ...n$.

The standard n-norm on X, is defined as:

$$\|x_1, x_2, ..., x_n\|_E = \begin{vmatrix} \langle x_1, x_1 \rangle & . & . & . & \langle x_1, x_n \rangle \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \langle x_1, x_1 \rangle & . & . & . & \langle x_1, x_1 \rangle \end{vmatrix}^{\frac{1}{2}},$$

$\langle ., . \rangle$ denotes the inner product on X. If $X = \mathbb{R}^n$, then this $n-$norm is exactly the same as the Euclidean $n-$norm $\|x_1, x_2, ..., x_n\|_E$ mentioned earlier. For $n = 1$ this $n-$norm is the usual norm $\|x\| = \langle x_1, x_1 \rangle^{\frac{1}{2}}$.

A sequence (x_k) in an $n-$normed space $(X, \|\cdot, ..., \cdot\|)$ is said to converge to some $L \in \mathbb{K}$ if $\lim_{k \to \infty} \|x_k - L, z_1, ...z_{n-1}\| = 0$ for every $z_1, z_2, ...z_{n-1} \in X$.

A sequence (x_k)in an $n-$normed space $(X, \|\cdot, ..., \cdot\|)$ is said to be Cauchy if $\lim_{k,p \to \infty} \|x_k - x_p, z_1, ...z_{n-1}\| = 0$ for every $z_1, z_2, ...z_{n-1} \in X$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n- normed space is said to be an n-Banach space.

Example 1.5. *Let* $\mathcal{I} = \mathcal{I}_\delta$ *where* $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0,$ *then* $\mathcal{I}_\delta$ *is an admissible ideal in* $\mathbb{N}$ *where ideal convergence coincides with statistical convergence.*
Define the sequence (x_k) *in n-normed space* $(X, \|., ..., .\|$ *by*

$$x_k = \begin{cases} (0, ..., k), & k = i^2, i \in \mathbb{N}, \\ (0, ..., 0), & otherwise. \end{cases}$$

let $x = (0, ..., 0)$*. Then for every* $\epsilon > 0$ *and* $x_1, x_2, ..., x_{n-1} \in X$

$$\{k \in \mathbb{N} : \|x_k - x, x_1, x_2, ..., x_{n-1}\| \geq \epsilon\} \subset \{1, 4, 9, 16, ..., k^2\}.$$

Also we have that $\delta(\{k \in \mathbb{N} : \|x_k - x, x_1, x_2, ..., x_{n-1}\| \geq \epsilon\}) = 0$ *for every* $\epsilon > 0$.
This implies that $\mathcal{I} - \lim_{k \to \infty} \|x_k, x_1, x_2, ..., x_{n-1}\| = \|x, x_1, x_2, ..., x_{n-1}\|$ *but the sequence* (x_k) *is not convergent to* x.

2 $\mathcal{I}$-Statistical Convergence and $\mathcal{I}$-Statistically Cauchy Sequence in n-normed Space

Now we give some useful definitions and examples based on $\mathcal{I}$-Statistical convergence and $\mathcal{I}$-Statistically Cauchy sequence in n-normed space.

Definition 2.1. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. The sequence (x_k) of X is said to be $\mathcal{I}$-statistically convergent to ξ if for each $\epsilon > 0, \exists\ \delta > 0$ such that for all $z_i \in X, i = 2, 3, 4...n$ the set*

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|x_k - \xi, z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

or equivalently if for each $\epsilon > 0$

$$\delta_{\mathcal{I}}(A_n(\epsilon)) = \mathcal{I} - \lim \delta_n(A_n(\epsilon)) = 0$$

where $(A_n(\epsilon)) = \{k \leq n : \|x_k - \xi, z_2, z_3, ...z_n\| \geq \epsilon\}$ and $\delta_n(A_n(\epsilon)) = \frac{|A_n(\epsilon)|}{n}$. If (x_k) is $\mathcal{I}-$convergent to ξ then we write $\mathcal{I} - st - \lim_{k\to\infty} \|x_k - \xi, z_2, z_3, ...z_n\| = 0$ or $\mathcal{I} - st - \lim_{k\to\infty} \|x_k, z_2, z_3, ...z_n\| = \|\xi, z_2, z_3...z_n\|$. The number ξ is called the $\mathcal{I}-$ limit of the sequence (x_k).

Remark 2.2. *If (x_k) is a sequence in X and ξ is any element of X then the set*

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|x_k - \xi, z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \delta\right\} = \emptyset$$

since if $z_i = 0,\ for\ i = 2, 3, 4...n$ then $\|x_k - \xi, z_2, z_3, ...z_n\| = 0 \ngeq \epsilon$

Definition 2.3. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non-trivial ideal in $\mathbb{N}$. The sequence (x_k) of X is said to be $\mathcal{I}$-statistically cauchy sequence if for every $\epsilon > 0,\ \delta > 0$ and all nonzero $z_i \in X, i = 2, 3, 4...n$ there exists a number N, dependent on ϵ such that*

$$\delta_{\mathcal{I}}\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|x_k - x_{N(\epsilon)}, z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \delta\right\} = 0,$$

i.e for every non zero $z_i \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|x_k - x_{N(\epsilon)}, z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

3 Main Results

Theorem 3.1 Let x_k be a sequence in n-normed space $(X, \|., ..., .\|)$, $\mathcal{I}$ be an admissible ideal and $L, L' \in X$. For each $z_i \in X$, if $\mathcal{I}$-st-$\lim\limits_{k\to\infty} \|x_k, z_2, z_3, ...z_n\| = \|L, z_2, z_3, ...z_n\|$ and $\mathcal{I}$-st-$\lim\limits_{k\to\infty} \|x_k, z_2, z_3, ...z_n\| = \|L', z_2, z_3, ...z_n\|$ then $L = L'$.

Proof: Assume $L \neq L'$. Then $L - L' \neq 0$, so there exist nonzero $z_2, z_3, ...z_n \in X$, such that $L - L'$ and $z_2, z_3, ...z_n$ are linearly independent (such z_i exist as dimension of X, $d \geq n$). Therefore for every $\epsilon > 0$ and $\delta > 0$,

$$\frac{1}{n}|\{k \leq n : \|L - L', z_2, z_3, ...z_n\| \geq \epsilon\}| = 2\delta.$$

Now,

$$\frac{1}{n}|\{k \leq n : \|L - x_k + x_k - L', z_2, z_3, ...z_n\| \geq \epsilon\}| = 2\delta,$$

$$\frac{1}{n}|\{k \leq n : \|(L - x_k) + (x_k - L'), z_2, z_3, ...z_n\| \geq \epsilon\}| = 2\delta,$$

$$\frac{1}{n}|\{k \leq n : \|(x_k - L), z_2, z_3, ...z_n\| \geq \epsilon\}| + \frac{1}{n}|\{k \leq n : \|(x_k - L'), z_2, z_3, ...z_n\| \geq \epsilon\}| \geq 2\delta,$$

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|(x_k - L), z_2, z_3, ...z_n\| \geq \epsilon\}| < \delta\right\}$$

$$\subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|(x_k - L^{'}), z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \delta, \right\}.$$

but $\delta_{\mathcal{I}} \left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|(x_k - L), z_2, z_3, ...z_n\| \geq \epsilon\}| < \delta \right\} = 0$ and hence contradicting the fact that $x_n \to L^{'}(\mathcal{I}\text{-st-lim})$.

Theorem 3.2 Let $\mathcal{I}$ be an admissible ideal. For each $z_i \in X$,
(i) If $\mathcal{I}\text{-st-lim}_{k\to\infty} \|x_k, z_2, z_3, ...z_n\| = \|x, z_2, z_3, ...z_n\|$ and $\mathcal{I}\text{-st-lim}_{k\to\infty} \|y_k, z_2, z_3, ...z_n\| = \|y, z_2, z_3, ...z_n\|$ then $\mathcal{I}\text{-st-lim}_{k\to\infty} \|x_k + y_k, z_2, z_3, ...z_n\| = \|x + y, z_2, z_3, ...z_n\|$
(ii) If $\mathcal{I}\text{-st-lim}_{k\to\infty} \|\alpha x_k, z_2, z_3, ...z_n\| = \|\alpha x, z_2, z_3, ...z_n\|$ for $\alpha \in \mathbb{R}$.

Proof:(i) Let $\mathcal{I}\text{-st-lim}_{k\to\infty} \|x_k, z_2, z_3, ...z_n\| = \|x, z_2, z_3, ...z_n\|$ and $\mathcal{I}\text{-st-lim}_{k\to\infty} \|y_k, z_2, z_3, ...z_n\| = \|y, z_2, z_3, ...z_n\|$ for every nonzero $z_i \in X$, then $\delta_{\mathcal{I}}(K_1) = 0$ and $\delta_{\mathcal{I}}(K_2) = 0$ where

$$K_1 = K_1(\epsilon) := \left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|x_k - x, z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \frac{\delta}{2} \right\}$$

and

$$K_2 = K_2(\epsilon) := \left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|y_k - y, z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \frac{\delta}{2} \right\}$$

for every $z_i \in X$. Let

$$K = K(\epsilon) := \left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|(x_k + y_k) - (x + y), z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \delta \right\}$$

To prove that $\delta_I(K) = 0$ it is sufficient to show that $K \subset K_1 \cup K_2$. Let $k_0 \in K$. Then

$$\frac{1}{n} |\{k \leq n : \|(x_{k_0} + y_{k_0}) - (x + y), z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \delta. \tag{3.1}$$

Suppose to the contrary that $k_0 \notin K_1 \cup K_2$. Then $k_0 \notin K_1$ and $k_0 \notin K_2$. This implies

$$\frac{1}{n} |\{k \leq n : \|x_{k_0} - x, z_2, z_3, ...z_n\| \geq \epsilon\}| < \frac{\delta}{2}.$$

$$\frac{1}{n} |\{k \leq n : \|y_{k_0} - y, z_2, z_3, ...z_n\| \geq \epsilon\}| < \frac{\delta}{2}.$$

Then, we get

$$\begin{aligned} \frac{1}{n} |\{k \leq n : \|(x_{k_0} + y_{k_0}) - (x + y), z_2, z_3, ...z_n\| &\leq \frac{1}{n} |\{k \leq n : \|x_{k_0} - x, z_2, z_3, ...z_n\| \\ &\quad + \frac{1}{n} |\{k \leq n : \|y_{k_0} - y, z_2, z_3, ...z_n\| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta, \end{aligned}$$

which contradicts (3.1). Hence $k_0 \in K_1 \cup K_2$, that is $K \subset K_1 \cup K_2$.

(ii) Let $\mathcal{I}\text{-st-lim}_{k\to\infty} \|x_k, z_2, z_3, ...z_n\| = \|x, z_2, z_3, ...z_n\|$, $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|x_k - x, z_2, z_3, ...z_n\| \geq \frac{\epsilon}{|\alpha|}\}| \geq \delta \right\} \in \mathcal{I}. \tag{3.2}$$

Then since $\|\alpha x_k - \alpha x, z_2, z_3, ...z_n\| = |\alpha|\|x_k - x, z_2, z_3, ...z_n\|$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|x_k - x, z_2, z_3, ...z_n\| \geq \frac{\epsilon}{|\alpha|}\}| \geq \delta\right\} \in \mathcal{I},$$

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |\alpha|\|x_k - x, z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I},$$

$$\left\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : \|\alpha x_k - \alpha x, z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

Hence from equation (3.2) we get $\mathcal{I}$-st-$\lim_{k\to\infty} \|ax_k, z_2, z_3, ...z_n\| = \|ax, z_2, z_3, ...z_n\|$, for every non zero $z_i \in X, i = 2, 3...n$.

Recall that we assume X to have dimension d, where $2 \leq n \leq d < \infty$, unless otherwise stated. Let $u = \{u_1, ..., u_(n-1)\}$ to be a basis for X. Then we have the following:

Theorem 3.3 Let $\mathcal{I}$ be an admissible ideal. A sequence $(x_k) \in X$ is $\mathcal{I}$-statistically convergent to $x \in X$ if and only if $\mathcal{I}$-st-$\lim\limits_{k\to\infty} \|x_k - x, \underbrace{u_i, u_i, ..., u_i}_{(n-1)}\| = 0$ for every $i = 1, ..., (n-1)$.

Proof: Let $(x_k) \in X$ is $\mathcal{I}$-statistically convergent to $x \in X$. Then by the definition of $\mathcal{I}$-st-convergence, we have

$$I - st - \lim_{k\to\infty} \|x_k - x, z_2, z_3, ..., z_n\| = 0$$

Then $\mathcal{I}$-st-$\lim\limits_{k\to\infty} \|x_k - x, \underbrace{u_i, u_i, ..., u_i}_{(n-1)times}\| = 0$ for every $i = 1, ..., (n-1)$, is trivial since every z_i can be expressed as a linear combination of u_i, for $i = 1, 2, 3....(n-1)$

Next we prove the result conversely.

Let us assume

$$\mathcal{I} - st - \lim_{k\to\infty} \|x_k - x, u_i, u_i, ..., u_i\| = 0 \quad for\ every \quad i = 1, ..., (n-1), \tag{3.3}$$

We want to show that

$$\left\{n \in N : \frac{1}{n}|\{k \leq n : \|x_k - x, z_2, z_3, ..., z_n\| \geq \epsilon\}| \geq \delta\right\} \tag{3.4}$$

For this, Let us consider the n-norm

$$\|x_k - x, z_2, z_3, ..., z_n\|$$

Also as $u = \{u_1, ..., u_{(n-1)}\}$ is a basis of X, then we have

$$z_2 = \sum_{i=1}^{n-1} \alpha_i^2 u_i, \quad z_3 = \sum_{i=1}^{n-1} \alpha_i^3 u_i, \qquad ... \qquad z_n = \sum_{i=1}^{n-1} \alpha_i^n u_i$$

This implies,

$$\|x_k - x, z_2, z_3, ..., z_n\| = \|x_k - x, \ \sum_{i=1}^{n-1} \alpha_i^2 u_i, \ \sum_{i=1}^{n-1} \alpha_i^3 u_i, ..., \sum_{i=1}^{n-1} \alpha_i^n u_i\|$$

As n is any positive integer, we have

$$\|x_k - x, z_2, z_3, ..., z_n\| \leq \|n(x_k - x), \ \sum_{i=1}^{n-1} \alpha_i^2 u_i, \ \sum_{i=1}^{n-1} \alpha_i^3 u_i, ..., \sum_{i=1}^{n-1} \alpha_i^n u_i\|$$

Using triangle inequality and distributing corresponding components for each z_i over n-norm we get

$$\begin{aligned}\|x_k - x, z_2, z_3, ..., z_n\| \leq \|(x_k - x),\ \underbrace{\alpha_1^2 u_1,\ \alpha_1^3 u_1, ..., \alpha_1^n u_1}_{(n-1)}\ \| \\ +\|(x_k - x),\ \underbrace{\alpha_2^2 u_2,\ \alpha_2^3 u_2, ..., \alpha_2^n u_2}_{(n-1)}\ \| + ... \\ +\|(x_k - x),\ \underbrace{\alpha_{(n-1)}^2 u_{(n-1)},\ \alpha_{(n-1)}^3 u_{(n-1)}, ..., \alpha_{(n-1)}^n u_{(n-1)}}_{(n-1)}\|.\end{aligned}$$

Let $\max \alpha_1^i = \alpha_1, \quad \forall i = 2, 3, 4...n.$, similarly let $\max \alpha_{(n-1)}^i = \alpha_{(n-1)}, \quad \forall i = 2, 3, 4, ..., n.$
Substituting these values in above equation, we get

$$\begin{aligned}\|x_k - x, z_2, z_3, ..., z_n\| &\leq \|(x_k - x),\ \alpha_1 u_1,\ \alpha_1 u_1, ..., \alpha_1 u_1\| + \|(x_k - x),\ \alpha_2 u_2,\ \alpha_2 u_2, ..., \alpha_2 u_2\| \\ &+ ... + \|(x_k - x),\ \alpha_{(n-1)} u_{(n-1)},\ \alpha_{(n-1)} u_{(n-1)}, ..., \alpha_{(n-1)} u_{(n-1)}\| \\ &\leq |\alpha_1|^{n-1}\|(x_k - x), u_1, u_1, ..., u_1\| + |\alpha_2|^{n-1}\|(x_k - x), u_2, u_2, ..., u_2\| \\ &+ ... + |\alpha_{(n-1)}|^{n-1}\|(x_k - x), u_{(n-1)}, u_{(n-1)}, ..., u_{(n-1)}\|\end{aligned}$$

By our assumption in equation (3.3) we have,

$\mathcal{I}\text{-st-}\lim_{k\to\infty} \|x_k - x, u_i, u_i, ..., u_i\| = 0$ for every $i = 1, ..., (n-1)$, which implies,
$\left\{n \in N : \frac{1}{n}\left|\{k \leq n : \{|\alpha_1|^{n-1}\|(x_k - x), u_1, u_1, ..., u_1\| + |\alpha_2|^{n-1}\|(x_k - x), u_2, u_2, ..., u_2\| + ... + |\alpha_{(n-1)}|^{n-1}\|(x_k - x), u_{(n-1)}, u_{(n-1)}, ..., u_{(n-1)}\|\} \geq \epsilon^{(n-1)}.\}\right| \geq \delta\right\}$
Hence we have

$$\begin{aligned}&\left\{n \in N : \frac{1}{n}|\{k \leq n : \|x_k - x, z_2, z_3, ..., z_n\| \geq \epsilon\}| \geq \delta\right\} \subset \\ &\left\{n \in N : \frac{1}{n}\left|\{k \leq n : \{\|(x_k - x), u_1, u_1, ..., u_1\| \geq \frac{\epsilon^{n-1}}{|\alpha_1|^{n-1}}\}\right| \geq \delta\right\} \\ &\cup\left\{n \in N : \frac{1}{n}\left|\{k \leq n : \{\|(x_k - x), u_2, u_2, ..., u_2\| \geq \frac{\epsilon^{n-1}}{|\alpha_2|^{n-1}}\}\right| \geq \delta\right\} \\ &\cup...\cup\left\{n \in N : \frac{1}{n}\left|\{k \leq n : \{\|(x_k - x), u_{(n-1)}, u_{(n-1)}, ..., u_{(n-1)}\|\} \geq \frac{\epsilon^{n-1}}{|\alpha_{(n-1)}|^{n-1}}\}\right| \geq \delta\right\}\end{aligned}$$

which gives,

$$\begin{aligned}&\left\{n \in N : \frac{1}{n}|\{k \leq n : \|x_k - x, z_2, z_3, ..., z_n\| \geq \epsilon\}| \geq \delta\right\} \subset \\ &\left\{n \in N : \frac{1}{n}\left|\{k \leq n : \{\|(x_k - x), u_1, u_1, ..., u_1\| \geq \frac{\epsilon}{|\alpha_1|}\}\right| \geq \delta\right\} \\ &\cup\left\{n \in N : \frac{1}{n}\left|\{k \leq n : \{\|(x_k - x), u_2, u_2, ..., u_2\| \geq \frac{\epsilon}{|\alpha_2|}\}\right| \geq \delta\right\}\end{aligned}$$

$$\cup ... \cup \left\{ n \in N : \frac{1}{n} \left| \{ k \leq n : \{ \| (x_k - x), u_{(n-1)}, u_{(n-1)}, ..., u_{(n-1)} \| \} \geq \frac{\epsilon}{|\alpha_{(n-1)}|} \} \right| \geq \delta \right\}$$

Since the right hand side of the above inclusion belongs to ideal, so does the left hand side. Consequently, we get $\mathcal{I}$-st-$\lim_{k\to\infty} \|x_k - x, z_2, z_3, ..., z_n\|$ for every nonzero $z_i \in X$. This proves the result.

Following theorem 3.3, we have the next lemma:

Lemma 3.4 Let I be an admissible ideal. A sequence $(x_k) \in X$ is I-statistically convergent to x in X if and only if $\mathcal{I}$-st-$\lim_{k\to\infty} max\|x_k - x, u_i, u_i....u_i\| = 0, \forall i = 1, ..., (n-1)$.

Definition 3.1. *In the light of lemma 3.4, we can define a norm on X, denoted by* $\|x\|_\infty$*, with respect to the basis* $u = u_1, ..., u_d$*, by*

$$\|x\|_\infty = \max\{\|x, \underbrace{u_i, u_i, ..., u_i}_{(n-1)}\| : i = 1, 2, ..., d\}$$

Using the derived norm $\|x\|_\infty$, Lemma 3.4 now reads:

Lemma 3.5 Let I be an admissible ideal. A sequence (x_k) in X is $\mathcal{I}$-statistically convergent to $x \in X$ if and only if $\mathcal{I}$-st-$\lim_{k\to\infty} \|x_k - x\|_\infty = 0$.

Associated to the derived norm $\|.\|_\infty$, we can define the balls $B(x, \epsilon)$ centered at x having radius ϵ by

$$B(x, \epsilon) := \{y : \|x - y\|_\infty \leq \epsilon\}$$

where $\|x - y\|_\infty = \max \|x - y, \underbrace{u_i, u_i, ..., u_i}_{(n-1)}\|$.

Using these balls, Lemma-3.5 becomes:

Lemma 3.6 Let I be an admissible ideal. A sequence (x_k) in X is I-statistically convergent to x in X if and only if $\delta_{\mathcal{I}}(A_n(\epsilon)) = 0$, where $A_n(\epsilon) = \{k \leq n : x_k \notin B(x, \epsilon)\}$.

Theorem 3.7 Any $\mathcal{I}$-st-Cauchy sequence (x_k) in an n-normed space $(X, \|., ..., .\|)$ is $\mathcal{I}$-st-convergent if and only if any $\mathcal{I}$-st-Cauchy sequence is $\mathcal{I}$-st-convergent with respect to $\|.\|_\infty$.

Proof: From previous result, it is clear that $\mathcal{I}$-st-convergence in n-norm is equivalent to that in $\|.\|_\infty$. That is for all $z_i \in X, i = 2, 3, ...n$

$$\mathcal{I} - st - \lim_{k\to\infty} \|x_k - x, z_2, ...z_n\| = 0$$

$$\Leftrightarrow \mathcal{I} - st - \lim_{k\to\infty} \|x_k - x\|_\infty = 0.$$

It is sufficient to show that (x_k) is $\mathcal{I}$-st-Cauchy sequence with respect to n-norm if and only if it is $\mathcal{I}$-st-Cauchy sequence with respect to $\|.\|_\infty$. Let (x_k) is $\mathcal{I}$-st-Cauchy sequence with respect to n-norm. Then there exists $N \in \mathbb{N}$ such that for $k, m \geq N$ we have

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \|x_k - x_m, z_2, z_3, ...z_n\| \geq \epsilon\}| \geq \delta \right\} \in \mathcal{I}$$

Consider,

$$\|x_k - x_m, z_2, z_3, ...z_n\| \geq \epsilon,$$

Then from lemma 3.3, we have $\|x_k - x_m, u_i, u_i, ...u_i\| \geq \epsilon$ for all $i = 1, 2...n$.
Hence max $\|x_k - x_m, u_i, u_i, ...u_i\| \geq \epsilon$ for all $i = 1, 2...n$.
By definition, it gives $\|x_k - x_m\|_\infty \geq \epsilon$.
Therefore (x_k) is $\mathcal{I}$-st-Cauchy with respect to $\|.\|_\infty$.

4 Conclusion

In this study, we develop the concept of Ideally-statistical convergence of sequences over n-normed space. Related algebraic and topological properties have been proved. The construction of max-norm of sequences over n-normed spaces lead to develop criterion for $\mathcal{I}$-st-Cauchy sequence to be $\mathcal{I}$-st-convergent.

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Competing interests

The authors declare that they have no competing interest.

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