The Fornberg-Whitham Equation Solved by the Differential Transform Method

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Authors' contributions

This work was carried out in collaboration between both authors. PAP formulated the problem and HN performed the analysis and computations. Both the authors have read and approved the final manuscript.

Original Research Article

Abstract

The Differential Transform Method is a powerful analytical method that can solve nonlinear partial differential equations. Yet, the method cannot be used to solve time-dependent partial differential equations that involve more than one partial derivative with respect to the temporal variable t when they are of the same order, as in the case of the Fornberg-Whitham type equations. In this paper, a new theorem is devised to overcome the aforementioned problem of the method, and it has been successfully applied to solve the Fornberg-Whitham equation. The other equations belonging to this group of equations, such as the Camassa-Holm equation and the Degasperi-Procesi equation, may also be solved by this approach.

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1   Introduction

The Differential Transform Method (DTM) has been proved to be an efficient tool to solve different classes of differential equations analytically. These include ordinary differential equations [1], classical partial differential equations (pde) such as the heat, wave and Poisson equations [2], the Sine-Gordon equation [3], the telegraph equation [4], the Fisher equation [5] and the Klein-Gordon equation [6]. In addition to pde-s, the method can be applied to delay differential equations as well. For example, in [7], the DTM is combined with the Laplace transform method to solve some nonlinear delay differential equations. Another class of differential equations solvable by the DTM are integro-differential equations, as can be seen in [8]. In this paper, examples of the said equation with proportional delays are solved. Adaptations of the method, such as the Fuzzy-Differential Transform Method [9] and the Fractional-Differential Transform Method [10] have been designed to solve the corresponding types of differential equations. In [10], the fractional differential transform method is used to solve fractional KdV equations, in addition to some third order nonlinear fractional pdes. The DTM can also solve systems of highly nonlinear pdes. Examples can be seen in ([11], [12]). The systems of nonlinear pdes are first converted to systems of ordinary differential equations with the help of a similarity variable and then solved by the DTM.

Yet, one very important pde, namely, the Fornberg-Whitham (FW) equation is found to be curiously missing in this list, even though the said equation has been solved by the Homotopy Analysis Method [13], the Variational Iteration Method [14], the Reduced Differential Transform Method [15] and the Homo-Separation of Variables Method [16]. On examination, it brought to light the difference of the nature of this equation with the others that are mentioned here. The FW equation has two first order partial derivatives with respect to the temporal variable, and this is seen to be the cause of the inability to solve the FW equation by the DTM. To circumvent this difficulty, a new theorem has been devised and successfully applied to solve the equation.

The following sections of this document discuss (a) the DTM briefly, (b) the FW equation, (c) the new theorem, and (d) the solution of the FW equation by the DTM. An analysis of how the theorem has facilitated this solution is also considered.

2   The Fornberg-Whitham Equation

The Korteweg de-Vries equation models unidirectional long water waves in shallow water surfaces [17]. The equation does not exhibit characteristics of short water waves such as the phenomena of the 'breaking of waves' and the existence of 'waves of greatest height' [18]. As a solution, in the year 1974, B. Fornberg and G.B. Whitham proposed the Fornberg-Whitham (FW) equation, the original form of which appeared in [18].

The FW equation is an integro-differential equation given by:

$$u_t + \alpha u u_x + \int_{-\infty}^{\infty} K(x - \xi) u_\xi(\xi, t) \, dt = 0, \tag{2.1}$$

where the kernal $K(x)$, when chosen suitably, produces the K-dV equation. The integral in equation (2.1) is removed by applying the operator given below:

$$\frac{\partial^2}{\partial x^2} - v^2,$$

where $v = \frac{\pi}{2}$, and (2.1) is transformed to a partial differential equation, which is:

$$\left(\frac{\partial^2}{\partial x^2} - v^2 \right) \left(u_t + \frac{3}{2} u u_x \right) + u_x = 0, \tag{2.2}$$

by taking $\alpha = \frac{3}{2}$ [17]. The limiting solution obtained by the authors in this paper is given by:

$$u = \frac{8}{9}e^{-\frac{1}{2}v\left(x-\frac{4}{3}t\right)}.$$ (2.3)

When v and α are both chosen to be one, and by applying the operator, equation (2.2) becomes the Fornberg-Whitham equation stated as follows:

$$u_t - u_{xxt} + u_x - uu_{xxx} + uu_x - 3u_xu_{xx} \quad = \quad 0.$$ (2.4)

This is the form of the FW equation solved by the Homotopy Analysis Method [13], the Variational Iteration Method [14], the Reduced Differential Transform Method [15] and the Homo-Separation of Variables Method [16]

3 The Differential Transform Method

The DTM was designed and used for the first time by Zhou in 1986. He applied the method to solve some pdes arising in problems in electricity [4]. The method is justified in its design, since the definition of the differential transforms of a function is formulated in such away that they generate the coefficients of the Taylor series expansion of the said function [19]. The definitions that follow explain this last statement.

3.1 Definition of the one dimensional differential transform

If $u(t)$ is analytic in the time domain T, the differential transform $U(k)$ of $u(t)$ is

$$U\left(k\right) = \frac{1}{k!}\left[\frac{d^ku\left(t\right)}{dt^k}\right]_{t=0}$$ (3.1)

where k is a non-negative integer [4].

3.1.1 Definition of the one dimensional inverse differential transform

If $u(t)$ is analytic in the time domain T, and (3.1) is the differential transform of $u(t)$, the inverse differential transform is given by [4]:

$$\begin{aligned} u\left(t\right) &= \sum_{k=0}^{\infty} U\left(k\right)t^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^ku\left(t\right)}{dt^k}\right]_{t=0}t^k. \end{aligned}$$ (3.2)

3.2 Definition of the two dimensional differential transform

If $w(x,y)$ is analytic and continuously differentiable at $(0,0)$, the differential transform of $w(x,y)$ is given by:

$$W\left(k,h\right) \quad = \quad \frac{1}{k!h!}\left[\frac{\partial^{k+h}w}{\partial x^k \partial y^h}\right]_{(0,0)},$$ (3.3)

where k and h are non-negative integers [20].

3.2.1   Definition of the inverse differential transform

The inverse differential transform of $W(k,h)$ is the function $w(x,y)$ itself, and is given by [20]:

$$w(x,y) \quad = \quad \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k,h) x^k y^h. \tag{3.4}$$

3.3   Important theorems for two dimensional differential transform

The following theorems from [20] are required to determine the differential transform of sums and products of functions, as well as of partial derivatives of these functions. All of them are deducible directly from definition (3.3).

Theorem 3.3.1. *If* $w(x,y) = u(x,y) \pm v(x,y)$, $W(k,h) = U(k,h) \pm V(k,h)$.

Theorem 3.3.2. *If* $w(x,y) = \lambda u(x,y)$, $\lambda \in R$, $W(k,h) = \lambda U(k,h)$.

Theorem 3.3.3. *If* $w(x,y) = u_x$, $W(k,h) = (k+1)U(k+1,h)$.

Theorem 3.3.4. *If* $w(x,y) = u_y$, $W(k,h) = (h+1)U(k,h+1)$.

Theorem 3.3.5. *If* $w(x,y) = \dfrac{\partial^{r+s} u}{\partial x^r \partial y^s}$,
$W(k,h) = (k+1)(k+2) \cdot \ldots (k+r)(h+1)(h+2) \cdot \ldots (h+s) U(k+r,h+s)$.

Theorem 3.3.6. *If* $w(x,y) = u(x,y)v(x,y)$,
$W(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} U(r,h-s) V(k-r,s)$.

Theorem 3.3.7. *If* $w(x,y) = u_x v_x$,
$W(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (r+1)(k-r+1) U(r+1,h-s) V(k-r+1,s)$.

Theorem 3.3.8. *If* $w(x,y) = u_y v_y$,
$W(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (s+1)(h-s+1) U(r,h-s+1) V(k-r,s+1)$.

The following theorems may be inferred from theorems (3.3.5) and (3.3.6).

Theorem 3.3.9. *If* $w(x,t) = uu_x$,
$W(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (k-r+1) U(r,h-s) U(k-r+1,s)$.

Theorem 3.3.10. *If* $w(x,t) = uu_{xxx}$,
$W(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (k-r+3)(k-r+2)(k-r+1) U(r,h-s) U(k-r+3,s)$.

Theorem 3.3.11. *If* $w(x,t) = u_x u_{xx}$,

$$W(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} (r+1)(k-r+2)(k-r+1) U(r+1,h-s) U(k-r+2,s).$$

The new theorem that is devised to solve the FW equation is stated below with its proof.

Theorem 3.3.12. *Suppose the function* $u(x,t)$ *of the two independent variables* x *and* t *is separable, so that*

$$u(x,t) = f(x)g(t).$$

Also, suppose that,

$$u(x,0) = f(x).$$

Then, for any pair of non-negative integers (k, h),

$$U(k, h) = \frac{1}{f(0)} U(k, 0) U(0, h), f(0) \neq 0, \tag{3.5}$$

where $U(k, h)$ is the (two-dimensional) differential transform of $u(x, t)$.

Proof

If $f(x)$ and $g(t)$ are continuously differentiable functions, their Taylor series expansions about the origin are given by:

$$f(x) \quad = \quad \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k, \text{ and } g(t) = \sum_{h=0}^{\infty} \frac{g^{(h)}(0)}{h!} t^h. \tag{3.6}$$

Let $F(k)$ and $G(h)$ represent the (one-dimensional) differential transforms of $f(x)$ and $g(t)$ respectively. Then, applying definition (3.1), $F(k)$ and $G(h)$ may be represented as follows:

$$F(k) \quad = \quad \frac{1}{k!} f^{(k)}(0), \text{ and } G(h) = \frac{1}{h!} g^{(h)}(0). \tag{3.7}$$

Hence,

$$
\begin{aligned}
u(x, t) \quad &= \quad f(x) g(t), \\
&= \quad \left\{ \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \right\} \times \left\{ \sum_{h=0}^{\infty} \frac{g^{(h)}(0)}{h!} t^h \right\} \\
&= \quad \left\{ \sum_{k=0}^{\infty} F(k) x^k \right\} \times \left\{ \sum_{h=0}^{\infty} G(h) t^h \right\} \\
&= \quad \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} F(k) G(h) \ x^k \ t^h.
\end{aligned}
\tag{3.8}
$$

Using definition (3.4), $u(x, t)$ may also be represented as stated below:

$$u(x, t) \quad = \quad \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U(k, h) \ x^k \ t^h, \tag{3.9}$$

producing a representation of $U(k, h)$ given by:

$$U(k, h) \quad = \quad F(k) G(h). \tag{3.10}$$

So, all that is left to be done in order to prove the theorem, is to establish the relations between the pairs $U(k, 0)$ and $F(k)$ and $U(0, h)$ and $G(h)$. For this purpose, consider the condition that $u(x, 0) = f(x)$. This implies that,

$$
\begin{aligned}
U(k, 0) \quad &= \quad \frac{1}{0! k!} \left[\frac{\partial^k u}{\partial x^k} \right]_{(0,0)} = \frac{1}{k!} \left[\frac{d^k f}{dx^k} \right]_{x=0} \\
&= \quad \frac{1}{k!} f^{(k)}(0) = F(k).
\end{aligned}
\tag{3.11}
$$

Using the definition of the two-dimensional differential transform once again,

$$
\begin{aligned}
U(0, h) \quad &= \quad \frac{1}{0! h!} \left[\frac{\partial^h u}{\partial t^h} \right]_{(0,0)} = \frac{1}{h!} \left[f(x) \frac{d^h g}{dt^h} \right]_{(0,0)} \\
&= \quad f(0) \times \frac{1}{h!} g^{(h)}(0) \\
&= \quad f(0) \times G(h).
\end{aligned}
\tag{3.12}
$$

Hence, it is found that,

$$
\begin{aligned}
U(k,h) &= F(k)G(h) = [U(k,0)]\left[\frac{U(0,h)}{f(0)}\right] \\
&= \frac{1}{f(0)}U(k,0)U(0,h).
\end{aligned}
$$

4 Solution of the Fornberg-Whitham Equation by the Differential Transform Method

The Fornberg-Whitham equation is once again stated as follows:

$$u_t - u_{xxt} + u_x - uu_{xxx} + uu_x - 3u_xu_{xx} = 0. \tag{4.1}$$

The initial condition associated with FW equation (4.1) is given by:

$$u(x,0) = e^{x/2}. \tag{4.2}$$

The solution of equation (4.1) together with the initial condition (4.2), as given in ([13], [16], [15]) is,

$$u(x,t) = e^{\left(\frac{x}{2} - \frac{2t}{3}\right)} \tag{4.3}$$

The initial stage of the method is to determine the differential transform of initial condition (4.2). From the Taylor series expansion of this function, its differential transform is obtained as:

$$U(k,0) = \frac{1}{k!}\left(\frac{1}{2}\right)^k. \tag{4.4}$$

The next stage is to determine the differential transform of FW equation (4.1), which may be rewritten as follows:

$$u_t - u_{xxt} = -u_x + uu_{xxx} - uu_x + 3u_xu_{xx}. \tag{4.5}$$

Applying theorems (3.3.3), (3.3.4), (3.3.9), (3.3.10) and (3.3.11), the differential transform of expression (4.5) is determined to be:

$$
\begin{aligned}
&(h+1)U(k,h+1) - (k+2)(k+1)(h+1)U(k+2,h+1) \\
={}& -(k+1)U(k+1,h) \\
&+ \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+3)(k-r+2)(k-r+1)U(r,h-s)U(k-r+3,s) \\
&- \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s) \\
&+ 3\sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+2)(k-r+1)U(r+1,h-s)U(k-r+2,s). \tag{4.6}
\end{aligned}
$$

Set $h = 0$ in the recurrence relation (4.6), to obtain the equation given by:

$$
\begin{aligned}
&U(k,1) - (k+2)(k+1)U(k+2,1) \\
={}& -(k+1)U(k+1,0) \\
&+ \sum_{r=0}^{k}(k-r+3)(k-r+2)(k-r+1)U(r,0)U(k-r+3,0) \\
&- \sum_{r=0}^{k}(k-r+1)U(r,0)U(k-r+1,0) \\
&+ 3\sum_{r=0}^{k}(r+1)(k-r+2)(k-r+1)U(r+1,0)U(k-r+2,0). \tag{4.7}
\end{aligned}
$$

Substitutions and simplifications reduce equation (4.7) to the following form:

$$U\left(k,1\right)-\left(k+2\right)\left(k+1\right)U\left(k+2,1\right)$$

$$= -\frac{1}{k!}\left(\frac{1}{2}\right)^{k+1}+\sum_{r=0}^{k}\frac{1}{r!}\frac{1}{\left(k-r\right)!}\left(\frac{1}{2}\right)^{k+3}$$

$$-\sum_{r=0}^{k}\frac{1}{r!}\frac{1}{\left(k-r\right)!}\left(\frac{1}{2}\right)^{k+1}+3\sum_{r=0}^{k}\frac{1}{r!}\frac{1}{\left(k-r\right)!}\left(\frac{1}{2}\right)^{k+3}$$

$$= -\frac{1}{k!}\left(\frac{1}{2}\right)^{k+1}+4\sum_{r=0}^{k}\frac{1}{r!}\frac{1}{\left(k-r\right)!}\left(\frac{1}{2}\right)^{k+3}-\sum_{r=0}^{k}\frac{1}{r!}\frac{1}{\left(k-r\right)!}\left(\frac{1}{2}\right)^{k+1}$$

$$= -\frac{1}{k!}\left(\frac{1}{2}\right)^{k+1}+4\left(\frac{1}{2}\right)^{k+3}\frac{2^{k}}{k!}-\left(\frac{1}{2}\right)^{k+1}\frac{2^{k}}{k!}, \tag{4.8}$$

since,

$$\sum_{r=0}^{k}\frac{1}{r!}\frac{1}{\left(k-r\right)!}=\frac{2^{k}}{k!},$$

The above expression thus becomes the compact form shown below:

$$U\left(k,1\right)-\left(k+2\right)\left(k+1\right)U\left(k+2,1\right)$$

$$= \frac{1}{k!}\left\{-\left(\frac{1}{2}\right)^{k+1}-\frac{1}{2}+\left(4\right)\left(\frac{1}{8}\right)\right\}=-\left(\frac{1}{2}\right)^{k+1}\frac{1}{k!}. \tag{4.9}$$

At this stage, it is possible to apply Theorem (3.3.12). Assuming that $U\left(0,1\right)\neq0$,

$$U\left(k,1\right)-\left(k+2\right)\left(k+1\right)U\left(k+2,1\right)$$

$$= U\left(k,0\right)U\left(0,1\right)-\left(k+2\right)\left(k+1\right)U\left(k+2,0\right)U\left(0,1\right)$$

$$= U\left(0,1\right)\left\{\frac{1}{k!}\left(\frac{1}{2}\right)^{k}-\left(k+2\right)\left(k+1\right)\frac{1}{\left(k+2\right)!}\left(\frac{1}{2}\right)^{k+2}\right\}$$

$$= U\left(0,1\right)\cdot\frac{1}{k!}\cdot\left(\frac{1}{2}\right)^{k}\cdot\frac{3}{4}=-\left(\frac{1}{2}\right)^{k+1}\frac{1}{k!}, \tag{4.10}$$

from equation (4.9). Simplifying the above,

$$U\left(0,1\right) = -\left(\frac{1}{2}\right)^{k+1}\left(2^{k}\right)\left(\frac{4}{3}\right)=-\left(\frac{1}{2}\right)\left(\frac{4}{3}\right)=-\frac{2}{3}. \tag{4.11}$$

$U\left(k,1\right)$ may now be found as shown below:

$$U\left(k,1\right) = U\left(k,0\right)U\left(0,1\right)=\left(\frac{1}{2}\right)^{k}\left(\frac{1}{k!}\right)\left(-\frac{2}{3}\right). \tag{4.12}$$

To evaluate $U\left(k,2\right)$ set $h=1$ in the transform (4.6), to obtain:

$$\left(2\right)U\left(k,2\right)-\left(k+2\right)\left(k+1\right)\left(2\right)U\left(k+2,2\right)$$

$$= -\left(k+1\right)U\left(k+1,1\right)$$

$$+\sum_{r=0}^{k}\sum_{s=0}^{1}\left(k-r+3\right)\left(k-r+2\right)\left(k-r+1\right)U\left(r,1-s\right)U\left(k-r+3,s\right)$$

$$-\sum_{r=0}^{k}\sum_{s=0}^{1}\left(k-r+1\right)U\left(r,1-s\right)U\left(k-r+1,s\right)$$

$$+3\sum_{r=0}^{k}\sum_{s=0}^{1}\left(r+1\right)\left(k-r+2\right)\left(k-r+1\right)U\left(r+1,1-s\right)U\left(k-r+2,s\right). \tag{4.13}$$

Executing the summation with respect to s, and making substitutions, the above equation becomes:

$$(2)U(k,2) - (k+2)(k+1)(2)U(k+2,2)$$

$$= \left(\frac{1}{k!}\right)\left(\frac{1}{2}\right)^{k+1}\left(\frac{2}{3}\right) + \sum_{r=0}^{k}(k-r+3)(k-r+2)(k-r+1) \times$$

$$\left\{\frac{-1}{r!}\left(\frac{1}{2}\right)^r\left(\frac{2}{3}\right)\frac{1}{(k-r+3)!}\left(\frac{1}{2}\right)^{k-r+3} + \frac{1}{r!}\left(\frac{1}{2}\right)^r\frac{-1}{(k-r+3)!}\left(\frac{1}{2}\right)^{k-r+3}\left(\frac{2}{3}\right)\right\}$$

$$- \sum_{r=0}^{k}(k-r+1)\left(\frac{1}{2}\right)^r\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^{k-r+1}\left\{\frac{-1}{r!}\frac{1}{(k-r+1)!} + \frac{1}{r!}\frac{-1}{(k-r+1)!}\right\}$$

$$+ 3\sum_{r=0}^{k}(r+1)(k-r+2)(k-r+1) \times$$

$$\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^{r+1}\left(\frac{1}{2}\right)^{k-r+2}\left\{\frac{-1}{(r+1)!}\frac{1}{(k-r+2)!} + \frac{1}{(r+1)!}\frac{-1}{(k-r+2)!}\right\}. \tag{4.14}$$

Simplify the expression that is obtained above to find that:

$$(2)U(k,2) - (k+2)(k+1)(2)U(k+2,2)$$

$$= \left(\frac{1}{k!}\right)\left(\frac{1}{2}\right)^{k+1}\left(\frac{2}{3}\right)$$

$$+ \left(\frac{2}{3}\right)\sum_{r=0}^{k}\left\{\frac{1}{r!(k-r)!}\right\}\left\{(-2)\left(\frac{1}{2}\right)^{k+3} + (2)\left(\frac{1}{2}\right)^{k+1} - (3)(2)\left(\frac{1}{2}\right)^{k+3}\right\}$$

$$= \left(\frac{1}{k!}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^{k+1}, \tag{4.15}$$

which, when simplified once again, reduces to the following expression:

$$2\{U(k,0)U(0,2) - (k+2)(k+1)U(k+2,0)U(0,2)\}$$

$$= 2U(0,2)\{U(k,0) - (k+2)(k+1)U(k+2,0)\}$$

$$= 2U(0,2)\left\{\frac{1}{k!}\left(\frac{1}{2}\right)^k - \frac{(k+2)(k+1)}{(k+2)!}\left(\frac{1}{2}\right)^{k+2}\right\}$$

$$= U(0,2)\left(\frac{1}{k!}\right)\left(\frac{1}{2}\right)^{k+1}(3). \tag{4.16}$$

Combine equations (4.15) and (4.16) to obtain:

$$U(0,2)\left(\frac{1}{k!}\right)\left(\frac{1}{2}\right)^{k+1}(3) = \left(\frac{1}{k!}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^{k+1}. \tag{4.17}$$

Thus,

$$U(0,2) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \left(\frac{1}{2!}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \left(\frac{1}{2!}\right)\left(\frac{2}{3}\right)^2. \tag{4.18}$$

$U(0,2)$ may also be represented by the following expression:

$$U(0,2) = \frac{(-1)^2}{2!}\left(\frac{2}{3}\right)^2. \tag{4.19}$$

Using the value of $U(0,2)$ obtained in relation (4.19), $U(k,2)$ is determined and is given by:

$$U(k,2) = U(k,0)U(0,2) = \left\{\frac{1}{k!}\left(\frac{1}{2}\right)^k\right\}\left\{\frac{(-1)^2}{2!}\left(\frac{2}{3}\right)^2\right\}$$

$$= \frac{(-1)^2}{k!\,2!}\left(\frac{2}{3}\right)^2\left(\frac{1}{2}\right)^k. \tag{4.20}$$

$U(k, 3)$ is also evaluated in a similar manner, and it is found to be as given below:

$$U(k,3) \quad = \quad \frac{(-1)^3}{k!3!}\left(\frac{2}{3}\right)^3\left(\frac{1}{2}\right)^k. \tag{4.21}$$

Since $U(k, i)$, for $i = 0, 1, 2, 3$, have a similar pattern, the general form of $U(k, i)$, namely, $U(k, h)$ is deduced to be:

$$U(k,h) \quad = \quad \frac{(-1)^h}{k!h!}\left(\frac{2}{3}\right)^h\left(\frac{1}{2}\right)^k \tag{4.22}$$

Substituting $U(k, h)$, as obtained above, in the Inverse Differential Transform given by expression (3.4), the solution of the Fornberg-Whitham equation is determined as follows:

$$
\begin{aligned}
u(x,t) \quad &= \quad \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)\, x^k\, t^h \\
&= \quad \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{(-1)^h}{k!h!}\left(\frac{2}{3}\right)^h\left(\frac{1}{2}\right)^k x^k\, t^h \\
&= \quad \left\{\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{2}\right)^k x^k\right\}\left\{\sum_{h=0}^{\infty}\frac{(-1)^h}{h!}\left(\frac{2}{3}\right)^h t^h\right\} \\
&= \quad \left(e^{x/2}\right)\times\left(e^{-2t/3}\right) \\
&= \quad e^{\left(\frac{x}{2}-\frac{2t}{3}\right)}
\end{aligned}
\tag{4.23}
$$

This is the same solution obtained in ([13], [16], [15]).

5 Analysis of the Application of The New Theorem in the Solution of the Fornberg-Whitham Equation by the Differential Transform Method

The differential transform of a function $u(x, t)$ and all its partial derivatives are numerical quantities, as definition (3.3) shows. For this reason, differential transform of the FW equation (4.6) is an algebraic equation, and so requires algebraic methods of solution. To explain this statement, an analysis of equation (4.9) is carried out. This relation, which is obtained after a sequence of algebraic simplifications of equation (4.7), contains two unknown quantities, and so cannot be solved unless a second algebraic equation containing the same unknown quantities is accessible. Since such an equation is not available, other methods are needed to proceed further. It is in this situation that Theorem (3.3.12) proves useful in representing $U(k, 1)$ and $U(k + 2, 1)$ as,

$$U(k,1) \quad = \quad U(k,0)\, U(0,1),$$

and,

$$U(k+2,1) \quad = \quad U(k+2,0)\, U(0,1)$$

Since the values of $U(k, 0)$ and $U(k + 2, 0)$ are known, it is possible to convert equation (4.9) to an equation involving just one unknown quantity, namely, $U(0, 1)$. This paved the way for the determination of $U(k, 1)$. $U(k, 2)$, $U(k, 3)$, etc, are all determined by similar processes, which finally lead to the solution of the FW equation.

6 Conclusion

The key tool that facilitated the solution of the Fornberg-Whitham equation by the differential transform method is Theorem (3.3.12). This theorem is applicable because the solution function $u(x,t)$ is separable, that is, $u(x,t)$ is expressible as a product of a function of x and a function of t.

The approach adopted to solve the FW equation, of combining the differential transform method with Theorem (3.3.12), may be applied to solve similar equations such as the Camassa-Holm equation [21] and the Degasperis-Procesi equation [22], since each one of these equations involves two first order partial derivatives with respect to the temporal variable as the FW equation.

Competing Interests

Authors have declared that no competing interests exist.

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