Eigenvalues and Eigenvectors for 3×3 Symmetric Matrices: An Analytical Approach

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Authors' contributions

This work was carried out in collaboration between both authors. ABS worked on equation derivation, plotting and writing. He also worked on the literature search. TK worked on equation derivation, writing and editing. Both authors contributed to conceptualization of the work. Both authors read and approved the final manuscript.

Original Research Article

Abstract

Research problems are often modeled using sets of linear equations and presented as matrix equations. Eigenvalues and eigenvectors of those coupling matrices provide vital information about the dynamics/flow of the problems and so needs to be calculated accurately. Analytical solutions are advantageous over numerical solutions because numerical solutions are approximate in nature, whereas analytical solutions are exact. In many engineering problems, the dimension of the problem matrix is 3 and the matrix is symmetric. In this paper, the theory behind finding eigenvalues and eigenvectors for order 3×3 symmetric matrices is presented. This is followed by the development of analytical solutions for the eigenvalues and eigenvectors, depending on patterns of the sparsity of the matrix. The developed solutions are tested against some examples with numerical solutions.

1 Introduction

Physical systems can be expressed using sets of linear and nonlinear equations. In the linear/linearized models, the system behavior is explained with the help of eigenvalues and corresponding eigenvectors. A

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majority of systems are applicable to, or expressed in, three-dimensional space and can be modeled using linear or linearized equations. In these linear equations, the dependent and independent vectors are coupled using a matrix, a "coupling matrix." This coupling matrix has different names. In stress-strain relationships or Hooke's Law for elastic deformation, it is called a stiffness matrix or compliance matrix depending on if stress or strain is the independent quantity. The state of stress or strain at a point in a body (see Fig. 1) is expressed using a second-order tensor, or a matrix of order(dimension) 3×3. The eigenvalues of this matrix express the principal stresses or strains, and the eigenvectors represent the directions in the space of the corresponding eigenvalues [1, 2, 3]. In dynamical systems (linear or linearized), eigenvalues of the coupling matrix express and characterize the fixed points of the system [4]. Eigenvalues and eigenvectors also provide essential information to many other problems like matrix diagonalization [5], vibration analysis [6], chemical reactions [7], face recognition [8], electrical networks [9], Markov chain model [10], atomic orbitals [11] and more.
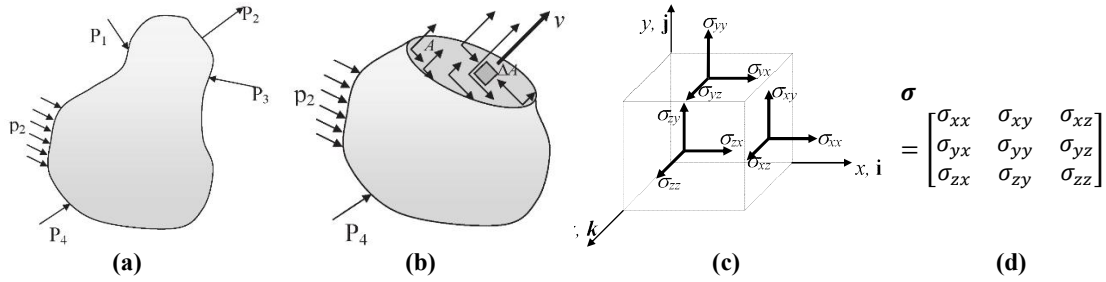

Fig. 1. (a) A general 3D body with different loads acting on its surface. (b) Free body diagram of a portion of (a). (c) Visualization of the stresses acting on different faces of a cube of vanishing dimensions representing a point in the body also called a *material point*. (d) The stress tensor at a material point [1]

Numerical methods work well for finding eigenvalues and eigenvectors with a certain accuracy. There are a number of efficient numerical algorithms available for finding eigenvalues and eigenvectors. Some prevalent algorithms are the Power iteration method [12], Jacobi method [13], and QR [14]. There are also efficient and publicly available software packages for finding eigenvalues and eigenvectors, i.e. LAPACK [15], GSL (GNU scientific library) [16], ARPACK [17], Armadillo [18], NumPy [19], SciPy [20] and Intel MKL [21]. These software are optimized for larger matrices, and using these software to find eigenvalues and eigenvectors of 3×3 symmetric matrices is not optimal. In this article, the authors propose an analytical routine to find the eigenvalues and eigenvectors efficiently in a simple and robust way. Any analytical solution is inherently more efficient since it is a straightforward solution utilizing substitution and thus requiring no iterations. The theory associated with each step of our algorithm is explained, and a couple of examples and a flowchart of an implemented code are shown.

2 Theory

Consider the following 3×3 real-valued symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \tag{0}$$

For any matrix of rank n (maximal number of independent columns or rows of a matrix), there exist n scalars λ_i for which eq (1) is true where $i = 1 \cdots n$. In the case herein of a 3×3 symmetric matrix $\mathbf{A}$, $n = 3$ and matrix symmetry means $a_{ij} = a_{ji}$.

$$\mathbf{A}\vec{v}_i = \lambda_i \vec{v}_i \tag{1}$$

The scalar quantity λ_i is termed the ith eigenvalue of the matrix, and the vector $\vec{v}_i$ the eigenvector of the matrix associated with λ_i. Re-writing eq. (1) and taking the determinant, the eigenvalues of matrix **A** can be found using eq. (2).

$$|\mathbf{A} - \lambda_i \mathbf{I}| = 0 \tag{2}$$

where $|\cdot|$ represents the determinant of any matrix, and **I** represents the identity matrix. The resulting equation from the determinant in eq. (2) is known as the characteristic polynomial [22] of matrix **A**. From elementary linear algebra, the eigenvalues of a real symmetric matrix **A** are also real [23]. Hence, the obtainment of the three real eigenvalues of a 3×3 symmetric matrix is tantamount to finding the roots of the cubic characteristic polynomial. The characteristic equation for this matrix can be written as:

$$\lambda^3 - \alpha\lambda^2 - \beta\lambda - \gamma = 0 \tag{3}$$

where,

$$\alpha = a_{11} + a_{22} + a_{33} \tag{4a}$$
$$\beta = a_{12}^2 + a_{13}^2 + a_{23}^2 - a_{11}a_{22} - a_{22}a_{33} - a_{33}a_{11} \tag{4b}$$
$$\gamma = a_{11}a_{22}a_{33} + 2a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{12}^2a_{33} - a_{13}^2a_{22} \tag{4c}$$

Here, α and γ are the trace and the determinant of matrix A, respectively. Alternatively, β can be written as $\left(-\begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} - \begin{vmatrix} a_{22} & a_{23} \\ a_{23} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{vmatrix}\right)$. In addition, α, β and γ are also known as the *invariants* of the matrix because they remain constant upon the rotational transformation of the matrix between different orthogonal coordinate systems. The characteristic equation can be solved analytically as follows [1] for the three roots λ_1, λ_2 and λ_3:

$$
\begin{aligned}
p &= -\left(\frac{3\beta + \alpha^2}{3}\right) \\
q &= -\left(\gamma + \frac{2\alpha^3}{27} + \frac{\alpha\beta}{3}\right) \\
\cos\phi &= -\frac{q}{2\sqrt{(|p|/3)^3}}
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\lambda_1 &= \frac{\alpha}{3} + 2\sqrt{\frac{|p|}{3}}\cos\left(\frac{\phi}{3}\right) \\
\lambda_2 &= \frac{\alpha}{3} - 2\sqrt{\frac{|p|}{3}}\cos\left(\frac{\phi - \pi}{3}\right) \\
\lambda_3 &= \frac{\alpha}{3} - 2\sqrt{\frac{|p|}{3}}\cos\left(\frac{\phi + \pi}{3}\right)
\end{aligned}
\tag{6}
$$

In some cases, and from the structure of matrix **A**, one can readily determine some eigenvalues. For example, knowing λ_1 and using the facts in eq. (7), one can derive the alternate eq. (8) to determine the other two eigenvalues λ_2 and λ_3

$$
\begin{aligned}
\lambda_1 + \lambda_2 + \lambda_3 &= \alpha \\
\lambda_1\lambda_2\lambda_3 &= \gamma
\end{aligned}
\tag{7}
$$

$$\lambda_{2,3} = \frac{\alpha - \lambda_1}{2} \pm \sqrt{\left(\frac{\alpha - \lambda_1}{2}\right)^2 - \frac{\gamma}{\lambda_1}} \tag{8}$$

Also, with knowledge of λ_1 and λ_2, one can find λ_3 using eqs. (7) and (9)

$$\lambda_3 = \alpha - \lambda_1 - \lambda_2 = \frac{\gamma}{\lambda_1 \lambda_2} \tag{9}$$

The eigenvectors of matrix **A** correspond to the eigenvalues as follows. Re-write eq. (1) as eq. (10):

$$(\mathbf{A} - \lambda_i \mathbf{I})\vec{v}_i = \vec{0} \tag{10}$$

Here, λ_i and $\vec{v}_i$ are the ith eigenvalue and i[th] eigenvector of **A**, where $i = 1 \cdots 3$.

Let $\vec{v}_i = \{l_i, m_i, n_i\}$. Eq. (10) can be written explicitly in full unabridged form as follows:

$$\begin{bmatrix} a_{11} - \lambda_i & a_{12} & a_{13} \\ a_{12} & a_{22} - \lambda_i & a_{23} \\ a_{13} & a_{23} & a_{33} - \lambda_i \end{bmatrix} \begin{bmatrix} l_i \\ m_i \\ n_i \end{bmatrix} = \mathbf{0} \tag{11}$$

Let $\mathbf{B}^i = \mathbf{A} - \lambda_i \mathbf{I}$. Eq. (11) can then be written as three linear algebraic equations as:

$$b_{11}^i l_i + b_{12} m_i + b_{13} n_i = 0 \tag{12a}$$

$$b_{12} l_i + b_{22}^i m_i + b_{23} n_i = 0 \tag{12b}$$

$$b_{13} l_i + b_{23} m_i + b_{33}^i n_i = 0 \tag{12c}$$

Notice in these last equations that $b_{st}^i = a_{st} - \lambda_i \delta_{st}$, where δ_{st} is the st[th] element of the Kronecker delta. Note that also for simplicity, the superscript i is dropped from b_{st}^i if s≠t. Since the equations (12a-12c) are not all linearly independent from one other, they cannot be solely used to find the unknowns l_i, m_i, and n_i. Hence, an additional or auxiliary equation is needed. This equation comes from the fact that the eigenvector $\vec{v}_i$ is a directional unit vector. With this fact, the additional equation is:

$$l_i^2 + m_i^2 + n_i^2 = 1 \tag{13}$$

In this paper, two eigenvectors are found by solving equations (12) and (13) for each of the two eigenvalues. Once these two vectors are found, the third eigenvector is determined from the cross product of the found two. Alternatively, the third eigenvector can also be found using equations (12) and (13).

As mentioned earlier, it is possible in some cases to readily determine the eigenvalues of matrix **A**. Hence, in order to make the current algorithm efficient computationally, several cases of matrix **A** are considered herein depending on the structure of the matrix.

3 Methodology

Case 1: Only Diagonal Elements Present (in *A*)

In the case that all off-diagonal elements in **A** are zero, the eigenvalues of the diagonal matrix are the diagonal elements [23], i.e. a_{11}, a_{22} and a_{33}. Here, and the eigenvectors are also simple, i.e. $\{\{1,0,0\},\{0,1,0\},\{0,0,1\}\}$

Case 2: Generalized Plane Stress or Plane Strain

Here, three sub-cases are considered:

Case 2.1: If $\mathbf{A} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{bmatrix}$

Here, and straight-forwardly, $\lambda_1 = a_{11}$. The other two eigenvalues λ_2 and λ_3 can be found either from eq. (6) or eq. (8). Alternatively, Mohr's circle in the *yz*-plane can be used to find λ_2 and λ_3 as below [1]. Hence, the three eigenvalues are:

$$\lambda_1 = a_{11}$$

$$\lambda_{2,3} = \frac{a_{22} + a_{33}}{2} \pm \sqrt{\left(\frac{a_{22} - a_{33}}{2}\right)^2 + a_{23}^2}$$

One eigenvector $\{1,0,0\}$ corresponding to $\lambda_1 = a_{11}$ is readily obtainable since all the off-diagonal terms in the first row and column (associated with the *x*-axis) are zeroes. The other two can be found in the *yz*-plane from equations (12) and (13). The last eigenvector can be obtained from the cross-product of the first two. These eigenvectors are:

$$\left\{ \{1,0,0\}, \left\{0, -\frac{b_{23}}{\sqrt{b_{22}^{i2}+b_{23}^2}}, \frac{b_{22}^i}{\sqrt{b_{22}^{i2}+b_{23}^2}}\right\}, \left\{0, -\frac{b_{22}^i}{\sqrt{b_{22}^{i2}+b_{23}^2}}, -\frac{b_{23}}{\sqrt{b_{22}^{i2}+b_{23}^2}}\right\} \right\}$$

Case 2.2: If $\mathbf{A} = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{13} & 0 & a_{33} \end{bmatrix}$

For this case, one eigenvalue ($\lambda_2 = a_{22}$) is readily obtainable and the corresponding eigenvector is also readily obtainable $\{0,1,0\}$. The other two eigenvalues (λ_1 and λ_3) are obtained from eq. (6) or (8), or using Mohr's circle as below:

$$\lambda_2 = a_{22}$$

$$\lambda_{1,3} = \frac{a_{11} + a_{33}}{2} \pm \sqrt{\left(\frac{a_{11} - a_{33}}{2}\right)^2 + a_{13}^2}$$

The eigenvectors are obtained from equations (12) and (13), and from doing a cross-product:

$$\left\{ \left\{-\frac{b_{13}}{\sqrt{b_{11}^{i2}+b_{13}^2}}, 0, \frac{b_{11}^i}{\sqrt{b_{11}^{i2}+b_{13}^2}}\right\}, \{0,1,0\}, \left\{\frac{b_{11}^i}{\sqrt{b_{11}^{i2}+b_{13}^2}}, 0, \frac{b_{13}}{\sqrt{b_{11}^{i2}+b_{13}^2}}\right\} \right\}$$

Case 2.3: If $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

This case follows similarly to Case 2.1 and Case 2.2, and hence the eigenvalues are:

$$\lambda_3 = a_{33}$$

$$\lambda_{1,2} = \frac{a_{11} + a_{22}}{2} \pm \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}^2}$$

and the eigenvectors are:

$$\left\{ \left\{ -\frac{b_{12}}{\sqrt{b_{11}^{i2} + b_{12}^2}}, \frac{b_{11}^i}{\sqrt{b_{11}^{i2} + b_{12}^2}}, 0 \right\}, \left\{ -\frac{b_{11}^i}{\sqrt{b_{11}^{i2} + b_{12}^2}}, -\frac{b_{12}}{\sqrt{b_{11}^{i2} + b_{12}^2}}, 0 \right\}, \{0,0,1\} \right\}$$

Case 3: General Cases

In general, there will be less than four off-diagonal zero elements in **A**. The eigenvalues can be found using eqs. (5) and (6). For the eigenvectors, 9 different cases exist that can be considered depending on the combination of equations resulting from the elimination of variables in eqs. (12a-12c). The eigenvectors are provided here explicitly in order to make the algorithm computationally efficient and robust.

For the sake of brevity, only three cases are shown here in detail.

Case 3.1: If $\left(b_{11}^i b_{23} - b_{13}b_{12}\right)b_{13} \neq 0$ *or* $\left(b_{12}^2 - b_{11}^i b_{22}^i\right)b_{13} \neq 0$

From $b_{12} \times (12a) - b_{11}^i \times (12b)$ one can write:

$$m_i = \frac{b_{11}^i b_{23} - b_{13}b_{12}}{b_{12}^2 - b_{11}^i b_{22}^i} n_i = \mathcal{Q}_i n_i \tag{14}$$

$$n_i = \frac{b_{12}^2 - b_{11}^i b_{22}^i}{b_{11}^i b_{23} - b_{13}b_{12}} m_i = \mathcal{R}_i m_i \tag{15}$$

From (14) and (12c) one can write:

$$l_i = -\frac{b_{23}\mathcal{Q}_i + b_{33}^i}{b_{13}} n_i = \mathcal{P}_i^n n_i \tag{16}$$

From (15) and (12c) one can write:

$$l_i = -\frac{b_{23} + b_{33}^i \mathcal{R}_i}{b_{13}} m_i = \mathcal{P}_i^m m_i \tag{17}$$

And using eq. (13) one can write:

$$n_i = \frac{1}{\sqrt{\mathcal{P}_i^{n2} + \mathcal{Q}_i^2 + 1}}$$

$$m_i = \frac{1}{\sqrt{\mathcal{P}_i^{m2} + 1 + \mathcal{R}_i^2}} \tag{18}$$

Hence, the eigenvector $\vec{v}_i$ corresponding to λ_i is either:

From (24) and (12a)

$$l_i = -\frac{b_{12}Q_i + b_{13}}{b_{11}^i}n_i = \mathcal{P}_i^n n_i \tag{26}$$

From (25) and (12a)

$$l_i = -\frac{b_{12} + b_{13}\mathcal{R}_i}{b_{11}^i}m_i = \mathcal{P}_i^m m_i \tag{27}$$

And using eq. (13) the following can be written

$$n_i = \frac{1}{\sqrt{\mathcal{P}_i^{n2} + Q_i^2 + 1}}$$

$$m_i = \frac{1}{\sqrt{\mathcal{P}_i^{m2} + 1 + \mathcal{R}_i^2}} \tag{28}$$

The eigenvector $\vec{v}_i$ is either

$$\left\{\frac{\mathcal{P}_i^n}{\sqrt{\mathcal{P}_i^{n2}+Q_i^2+1}}, \frac{Q_i}{\sqrt{\mathcal{P}_i^{n2}+Q_i^2+1}}, \frac{1}{\sqrt{\mathcal{P}_i^{n2}+Q_i^2+1}}\right\} \text{ Or } \left\{\frac{\mathcal{P}_i^m}{\sqrt{\mathcal{P}_i^{m2}+1+\mathcal{R}_i^2}}, \frac{1}{\sqrt{\mathcal{P}_i^{m2}+1+\mathcal{R}_i^2}}, \frac{\mathcal{R}_i}{\sqrt{\mathcal{P}_i^{m2}+1+\mathcal{R}_i^2}}\right\}$$

Case 3.4*:* If $\left(b_{12}b_{23} - b_{13}b_{22}^i\right)b_{23} \neq 0$ **or** $\left(b_{11}^i b_{22}^i - b_{12}^2\right)b_{23} \neq 0$

The eigenvector is either

$$\left\{\frac{\mathcal{P}_i}{\sqrt{\mathcal{P}_i^2+Q_i^{n2}+1}}, \frac{Q_i^n}{\sqrt{\mathcal{P}_i^2+Q_i^{n2}+1}}, \frac{1}{\sqrt{\mathcal{P}_i^2+Q_i^{n2}+1}}\right\} \text{ Or } \left\{\frac{1}{\sqrt{1+Q_i^{l2}+\mathcal{R}_i^2}}, \frac{Q_i^l}{\sqrt{1+Q_i^{l2}+\mathcal{R}_i^2}}, \frac{\mathcal{R}_i}{\sqrt{1+Q_i^{l2}+\mathcal{R}_i^2}}\right\},$$

Where

$$\frac{b_{12}b_{23} - b_{13}b_{22}^i}{b_{11}^i b_{22}^i - b_{12}^2} = \mathcal{P}_i \text{ , } \frac{b_{11}^i b_{22}^i - b_{12}^2}{b_{12}b_{23} - b_{13}b_{22}^i} = \mathcal{R}_i \text{ , } -\frac{b_{13}\mathcal{P}_i + b_{33}^i}{b_{23}} = Q_i^n \text{ , and } -\frac{b_{13} + b_{33}^i\mathcal{R}_i}{b_{23}} = Q_i^l$$

Case 3.5: If $\left(b_{12}b_{33}^i - b_{13}b_{23}\right)b_{22}^i \neq 0$ or $\left(b_{11}^i b_{23} - b_{12}b_{13}\right)b_{22}^i \neq 0$

The eigenvector is either

$$\left\{\frac{\mathcal{P}_i}{\sqrt{\mathcal{P}_i^2+Q_i^{n2}+1}}, \frac{Q_i^n}{\sqrt{\mathcal{P}_i^2+Q_i^{n2}+1}}, \frac{1}{\sqrt{\mathcal{P}_i^2+Q_i^{n2}+1}}\right\} \text{ Or } \left\{\frac{1}{\sqrt{1+Q_i^{l2}+\mathcal{R}_i^2}}, \frac{Q_i^l}{\sqrt{1+Q_i^{l2}+\mathcal{R}_i^2}}, \frac{\mathcal{R}_i}{\sqrt{1+Q_i^{l2}+\mathcal{R}_i^2}}\right\},$$

Where

$$\frac{b_{12}b_{33}^i - b_{13}b_{23}}{b_{11}^i b_{23} - b_{12}b_{13}} = \mathcal{P}_i \text{ , } \frac{b_{11}^i b_{23} - b_{12}b_{13}}{b_{12}b_{33}^i - b_{13}b_{23}} = \mathcal{R}_i \text{ , } -\frac{b_{12}\mathcal{P}_i + b_{23}}{b_{22}^i} = Q_i^n \text{ , and } -\frac{b_{12} + b_{23}\mathcal{R}_i}{b_{22}^i} = Q_i^l$$

Case 3.6: If $\left(b_{22}^i b_{33}^i - b_{23}^2\right)b_{12} \neq 0$ or $\left(b_{12}b_{23} - b_{22}^i b_{13}\right)b_{12} \neq 0$

The eigenvector is either

$$\left\{\frac{\mathcal{P}_i}{\sqrt{\mathcal{P}_i^2 + Q_i^{n2} + 1}}, \frac{Q_i^n}{\sqrt{\mathcal{P}_i^2 + Q_i^{n2} + 1}}, \frac{1}{\sqrt{\mathcal{P}_i^2 + Q_i^{n2} + 1}}\right\} \text{ Or } \left\{\frac{1}{\sqrt{1 + Q_i^{l2} + \mathcal{R}_i^2}}, \frac{Q_i^l}{\sqrt{1 + Q_i^{l2} + \mathcal{R}_i^2}}, \frac{\mathcal{R}_i}{\sqrt{1 + Q_i^{l2} + \mathcal{R}_i^2}}\right\},$$

Where

$$\frac{b_{22}^i b_{33}^i - b_{23}^2}{b_{12}b_{23} - b_{22}^i b_{13}} = \mathcal{P}_i \ , \ \frac{b_{12}b_{23} - b_{22}^i b_{13}}{b_{22}^i b_{33}^i - b_{23}^2} = \mathcal{R}_i \ , \ -\frac{b_{11}^i \mathcal{P}_i + b_{13}}{b_{12}} = Q_i^n \ , \text{ and } -\frac{b_{11}^i + b_{13}\mathcal{R}_i}{b_{12}} = Q_i^l$$

Case 3.7: If $\left(b_{13}b_{22}^i - b_{12}b_{23}\right)b_{33}^i \neq 0$ or $\left(b_{11}^i b_{23} - b_{13}b_{12}\right)b_{33}^i \neq 0$

The eigenvector is either

$$\left\{\frac{\mathcal{P}_i}{\sqrt{\mathcal{P}_i^2 + 1 + \mathcal{R}_i^{m2}}}, \frac{1}{\sqrt{\mathcal{P}_i^2 + 1 + \mathcal{R}_i^{m2}}}, \frac{\mathcal{R}_i^m}{\sqrt{\mathcal{P}_i^2 + 1 + \mathcal{R}_i^{m2}}}\right\} \text{ Or } \left\{\frac{1}{\sqrt{1 + Q_i^2 + \mathcal{R}_i^{l2}}}, \frac{Q_i}{\sqrt{1 + Q_i^2 + \mathcal{R}_i^{l2}}}, \frac{\mathcal{R}_i^l}{\sqrt{1 + Q_i^2 + \mathcal{R}_i^{l2}}}\right\},$$

Where

$$\frac{b_{13}b_{22}^i - b_{12}b_{23}}{b_{11}^i b_{23} - b_{13}b_{12}} = \mathcal{P}_i \ , \ \frac{b_{11}^i b_{23} - b_{13}b_{12}}{b_{13}b_{22}^i - b_{12}b_{23}} = Q_i \ , \ -\frac{b_{13}\mathcal{P}_i + b_{23}}{b_{33}^i} = \mathcal{R}_i^m \ , \text{ and } -\frac{b_{13} + b_{23}Q_i}{b_{33}^i} = \mathcal{R}_i^l$$

Case 3.8: If $\left(b_{13}b_{23} - b_{12}b_{33}^i\right)b_{23} \neq 0$ or $\left(b_{11}^i b_{33}^i - b_{13}^2\right)b_{23} \neq 0$

The eigenvector is either

$$\left\{\frac{\mathcal{P}_i}{\sqrt{\mathcal{P}_i^2 + 1 + \mathcal{R}_i^{m2}}}, \frac{1}{\sqrt{\mathcal{P}_i^2 + 1 + \mathcal{R}_i^{m2}}}, \frac{\mathcal{R}_i^m}{\sqrt{\mathcal{P}_i^2 + 1 + \mathcal{R}_i^{m2}}}\right\} \text{ Or } \left\{\frac{1}{\sqrt{1 + Q_i^2 + \mathcal{R}_i^{l2}}}, \frac{Q_i}{\sqrt{1 + Q_i^2 + \mathcal{R}_i^{l2}}}, \frac{\mathcal{R}_i^l}{\sqrt{1 + Q_i^2 + \mathcal{R}_i^{l2}}}\right\},$$

Where

$$\frac{b_{13}b_{23} - b_{12}b_{33}^i}{b_{11}^i b_{33}^i - b_{13}^2} = \mathcal{P}_i \ , \ \frac{b_{11}^i b_{33}^i - b_{13}^2}{b_{13}b_{23} - b_{12}b_{33}^i} = Q_i \ , \ -\frac{b_{12}\mathcal{P}_i + b_{22}^i}{b_{23}} = \mathcal{R}_i^m \ , \ -\frac{b_{12} + b_{22}^i Q_i}{b_{23}} = \mathcal{R}_i^l$$

Case 3.9: If $\left(b_{23}^2 - b_{22}^i b_{33}^i\right)b_{13} \neq 0$ or $\left(b_{12}b_{33}^i - b_{23}b_{13}\right)b_{13} \neq 0$

The eigenvector is either

$$\left\{\frac{\mathcal{P}_i}{\sqrt{\mathcal{P}_i^2 + 1 + \mathcal{R}_i^{m2}}}, \frac{1}{\sqrt{\mathcal{P}_i^2 + 1 + \mathcal{R}_i^{m2}}}, \frac{\mathcal{R}_i^m}{\sqrt{\mathcal{P}_i^2 + 1 + \mathcal{R}_i^{m2}}}\right\} \text{ Or } \left\{\frac{1}{\sqrt{1 + Q_i^2 + \mathcal{R}_i^{l2}}}, \frac{Q_i}{\sqrt{1 + Q_i^2 + \mathcal{R}_i^{l2}}}, \frac{\mathcal{R}_i^l}{\sqrt{1 + Q_i^2 + \mathcal{R}_i^{l2}}}\right\},$$

where

$$\frac{b_{23}^2 - b_{22}^i b_{33}^i}{b_{12}b_{33}^i - b_{23}b_{13}} = \mathcal{P}_i \ , \ \frac{b_{12}b_{33}^i - b_{23}b_{13}}{b_{23}^2 - b_{22}^i b_{33}^i} = Q_i \ , \ -\frac{b_{11}^i \mathcal{P}_i + b_{12}}{b_{13}} = \mathcal{R}_i^m \ , \ -\frac{b_{11}^i + b_{12}Q_i}{b_{13}} = \mathcal{R}_i^l$$

For all Case 3 (Cases 3.1-3.9), since there is at least one non-zero shear component ($b_{st}^i \neq 0, s \neq t$) then according to Mohr's circle there will be at least two distinct principal stresses or eigenvalues, i.e. the multiplicity of the eigenvalue is maximum 2 in this case. This produces two distinct eigenvectors using the above methodology. Therefore, the third eigenvector can be obtained by cross product of the two distinct ones.

4 Results

Presented below is a couple of examples for different structures or sparsities of [A].

Example 1: Assume a piece of material/body under loading similar to what is shown in Fig. 1. The stress tensor, or stress state, at a point in the body is given by:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0.6375 & 0 & -1.7835 \\ 0 & 7.1568 & 0 \\ -1.7835 & 0 & 1.4508 \end{bmatrix} \text{MPa}$$

Fig. 2(a) graphically shows the stresses in the *xyz*-coordinate system. To find the principal stresses and their direction in space, this is tantamount to an eigenvalue and eigenvector problem, as presented above.
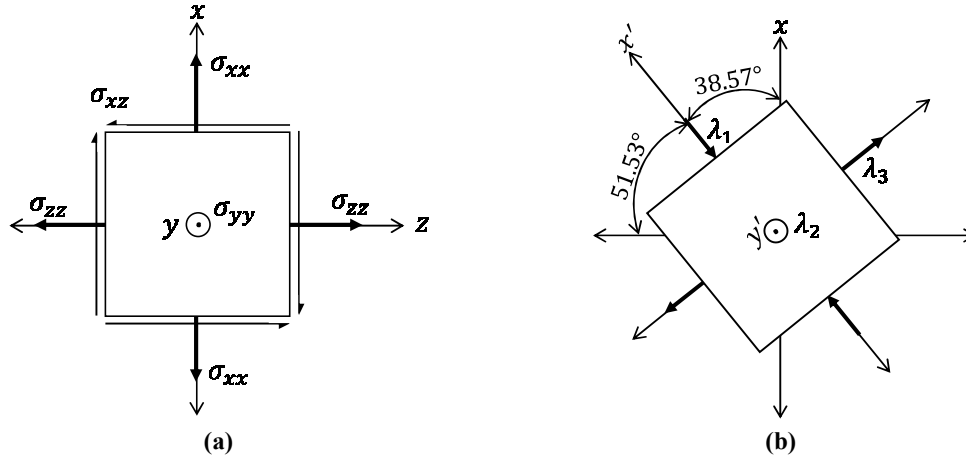

Fig. 2. Stresses at a material point expressed in the *xyz*-coordinate system. (b) Stresses at the same material point but expressed in the principal stress space (or rotated coordinate system) showing only normal stresses (principal stresses) and not shearing

First off, one needs to determine the case of eigenvalue/eigenvector problem to which this stress state belongs. Comparing with above, this belongs under Case 2.2 (Generalized Plane Stress or Plane Strain). Using the equations there, the calculated eigenvalues are {−0.7851, 7.1568, 2.8734} and the calculated eigenvectors are {{0.7818, 0, 0.6236}, {0, 1, 0}, {−0.6236, 0, 0.7818}}.

Note that each of the eigenvectors represent a vector in a 3×3 rotational transformation matrix, call it $\boldsymbol{\beta}$, that transforms tensorial quantities from the unprimed coordinate system in Fig. 2(a) to the primed coordinate system in Fig. 2(b), or vice versa. The primed coordinate system here is in line with the principal stress directions. In fact, the ij[th] element of $\boldsymbol{\beta}$ or β_{ij} is generally given by $\beta_{ij} = \vec{e}_i' \cdot \vec{e}_j = cos\theta_{ij}$, where $\vec{e}_i'$ represents one of the directional unit vectors along the positive primed coordinate system axes, and $\vec{e}_j$ represents one of the directional unit vectors along the positive unprimed coordinate system axes. The elements of $\boldsymbol{\beta}$ are thus the direction cosines between the coordinate axes of the primed and the unprimed coordinate systems [1].

Example 2: Assume the stress state at a different material point in the loaded body to be:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 0 & -1.0298 & 1.0792 \\ -1.0298 & 1.2554 & 0 \\ 1.0792 & 0 & 0.1547 \end{bmatrix} \text{MPa}$$

Again, to find the principal stresses and their directions, one needs to solve the eigenvalue/eigenvector problem. Comparing with above, this case belongs under Case 3: General Cases. More specifically, it falls under Case 3.1.

To find the eigenvalues, one needs to apply eqs. (2-6). Here, $= 1.4101$, $\beta = 2.03095$, $\gamma = -1.62619$, $p = -2.69374$, $q = 0.463883$, $\phi = 1.84689$ rad. Using these numbers, the three eigenvalues are $\{\lambda_1, \lambda_2, \lambda_3\} = \{-1.2514, 0.6442, 2.0172\}$.

To find the eigenvectors, one needs to use eqs. (14-18). Starting with the first eigenvalue λ_1, one can calculate $Q_i = -0.5352$, $\mathcal{R}_i = -1.8685$, $\mathcal{P}_i^n = -1.3029$, and $\mathcal{P}_i^m = 2.4344$. Using these numbers and eq. (18.1), it can be found that $n_i = 0.5789$. Using eq. (16), one can calculate $l_i = -0.7542$. Finally using eq. (14), one can calculate $m_i = -0.3098$. Now according to these numbers, the corresponding eigenvector is $\{-0.7542, -0.3098, 0.5789\}$.

Repeating the procedure above for the second eigenvalue λ_2, one gets: $Q_i = 0.7642$ and $\mathcal{R}_i = 1.3085$. Using eq. (16-17), $\mathcal{P}_i^n = 0.4536$ and $\mathcal{P}_i^m = 0.5935$ are calculated. Following this, one can calculate $n_i = 0.7475$, $l_i = 0.3390$ and $m_i = 0.5712$, which means that the corresponding eigenvector is $\{0.3390, 0.5713, 0.7475\}$.

To find the last eigenvector corresponding to the last eigenvalue λ_3, one can either repeat the above procedure similar to λ_1 and λ_2, or one can find it using a cross product between the first two eigenvectors. Both paths give the same result for the third eigenvector, i.e. $\{0.5622, -0.7602, 0.3257\}$.

It is important to note here that the numbers for Example 1 and Example 2 above were checked, in way of verification, against numerical computations by Matlab 2020a [24].

5 Algorithm

The eigenvalues are obtained using eqs. (4), (5), and (6). Alternatively, eqs. (8) or (9) or Mohr's circle can be invoked for obtaining the eigenvalues. After that, the case of matrix **A** should be determined per the cases above. Once this determination is made, the correct set of equations can be used to find at least two different eigenvectors. The third eigenvector can be obtained from the cross product of the first two eigenvectors. A flowchart describing the whole algorithm, used by the authors in coding, is shown in Fig. 3. A pseudocode is also presented in Fig. 4.
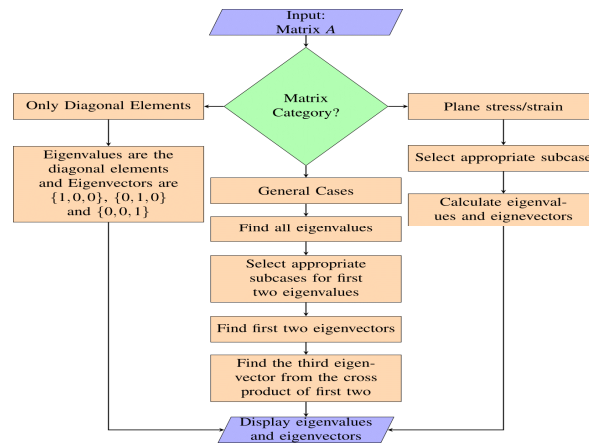

Fig. 3. Flowchart for coding eigenvalues and eigenvectors obtainment

```
Input matrix A
Determine Matrix category
IF only diagonal elements THEN
    Eigenvalues are A[1,1], A[2,2], A[3,3]
    Eigenvalues are 1,0,0, 0,1,0 and 0,0,1
ELSE IF Plane stress/strain THEN
    Determine subcase
    Find eigenvalues and eigenvectors using related equations
    for that case
ELSE General Cases
    FIND eigenvalues using equation 6
    For first two eigenvalues determine the subcases and find
    the eigenvectors as shown in the related subcase
    Find the third eigenvector from the cross product of first
    two eigenvectors
END IF
PRINT eigenvalues and eigenvectors
END
```

Fig. 4. Pseudocode for coding eigenvalues and eigenvectors obtainment

6 Conclusion

In this article, the theory behind and algorithm to find eigenvalues and eigenvectors of a 3×3 symmetric matrix were presented. The current algorithm presents an efficient approach to finding the eigenvalues and eigenvectors for symmetric matrices of order(dimension) 3×3 over publicly available software packages since they have a learning curve and some are difficult to use. The current method is simple, optimized, robust and can easily be implemented in any programming language or even in a handheld calculator. The presented analytical solutions can prove to be handy for applications requiring repetitive finding/calculations of eigenvalues and eigenvectors. The solutions also represent a valuable exercise into understanding linear algebra for learners in the field.

Competing Interests

Authors have declared that no competing interests exist.

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References

[1]   Khraishi TA, Shen Y-L. Introductory continuum mechanics with applications to elasticity, San Diego: Cognella; 2013.

[2]   Ugural AC, Fenster SK. Advanced strength and applied elasticity, London: Pearson Education; 2003.

[3]   Sadd MH. Elasticity: theory, applications, and numerics, Cambridge: Academic Press; 2009.

[4]   Thompson JMT, Stewart HB. Nonlinear dynamics and chaos, Hoboken: John Wiley & Sons; 2002.

[5]   Anton H, Rorres C. Elementary linear algebra: Applications Version, Hoboken: John Wiley; 2013.

[6]   Meirovitch L. Elements of vibration analysis, New York: McGraw-Hill Companies; 1975.

[7]   Marcus R. On the analytical mechanics of chemical reactions. Quantum mechanics of linear collisions. The Journal of Chemical Physics. 1966;45(12):4493-4499.

[8]   Turk MA, Pentland AP. Face recognition using eigenfaces. in IEEE, Maui; 1991.

[9]   Dorfler F, Bullo F. Kron reduction of graphs with applications to electrical networks. IEEE Trans. on Circuits and Systems.2013;60(1):150-163.

[10]   Karlin S. A first course in stochastic processes, Cambridge: Academic press; 2014.

[11]   Hehre WJ, Stewart RF, Pople JA. self-consistent molecular-orbital methods. i. use of gaussian expansions of Slater-type atomic orbitals. The Journal of Chemical Physics. 1969;51(6):2657-2664.

[12]   Chapra SC, Canale RP. Numerical methods for engineers, New York: Mcgraw-hill; 1998.

[13]   Mathias R. Accurate eigensystem computations by Jacobi methods. SIAM journal on matrix analysis and applications. 1995;16(3):977-1003.

[14]   Anley EF. The QR method for determining all eigenvalues of real square matrices. Pure and Applied Mathematics Journal. 2016;5(4):113-119.

[15]   Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J, Du Croz J, Greenbaum A, Hammarling S, McKenney A, Sorensen D. LAPACK Users' Guide, 3rd ed., Philadelphia: Society for Industrial and Applied Mathematics; 1999.

[16]   Galassi M. GNU Scientific Library Reference Manual, Online: Network Theory Ltd; 2016.

[17]   Lehoucq RB, Sorensen DC, Yang C. ARPACK users' guide: solution of large-scale eigenvalue problems with implicitly restarted Arnoldi methods, Online: Siam; 1998.

[18]   Sanderson C, Curtin R. Armadillo: a template-based C++ library for linear algebra. Journal of Open Source Software. 2016;1(2):26.

[19]   Oliphant TE. Guide to NumPy, Online: Trelgol Publishing USA; 2006.

[20]   Oliphant TE. Python for scientific computing. Computing in Science and Engg. 2007;9(3):10-20.

[21]   Zone ID. Reference Manual for Intel Math Kernel Library 11.2, Online: Intel; 2014.

[22]   Jeffrey A. Matrix operations for engineers and scientists: an essential guide in linear algebra, Berlin: Springer Science & Business Media; 2010.

[23]   Keshav S. Mathematical foundations of computer networking, Boston: Addison-Wesley; 2012.

[24]   Mathworks, Matlab 2020a, Natick; 2020.