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Boundary Value Method for Direct Solution of Sixth-Order Boundary Value Problems

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this work, 7^{th} order continuous block methods called the Boundary Value Method (BVM) for the numerical approximation of sixth-order boundary Value Problem (BVPs) is proposed. These methods are derived using the Chebyshev polynomial as basis functions. The BVM comprises the main methods and additional methods, put together to form a block method and thus solved simultaneously to obtain an approximate solution for sixth-order BVPs. This method do not require a starting value as it is self-starting. The BVM is found to be consistent and its convergence was discussed. Numerical examples are shown to illustrate the applicability of the method. To show the efficiency of this method, the approximated solution derived from the methods is compared to the exact solutions of the problem and thus maximum errors are recorded and compared to those in other method from literature.

Keywords: Sixth order boundary value problems; chebyshev polynomials; block methods.

AMS Subject Classification: 65L05, 65L06, 65L10, 65L12.

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1 Introduction

In this work, problem of the form

$$y^{(vi)}(x) = f(x, y(x), y'(x), \dots, y^{(v)})(x), \quad x \in [a, b] \tag{1}$$

subject to the boundary conditions:

$$\left.\begin{array}{ll} y(a) = a_0, & y(b) = a_1, \\ y'(a) = a_0, & y'(b) = b_1, \\ y''(a) = c_0, & y''(b) = c_1, \end{array}\right\} \tag{2}$$

are considered. Where f is Lipschitz on $[a, b]$, to ensure existence and uniqueness of the solution $y \in C^n[a, b]$, a_j, b_j, c_j $(j = 0,1)$ are finite real arbitrary constants.

Sixth-order Boundary Value Problems (BVPs) arise in astrophysics. Consider A-type stars which are believed to be surrounded by narrow convecting layers bounded by stable layers [1]. When an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. The governing ordinary differential equation is of sixth order and this instability is of ordinary convection [2,3]. These problems may be considered as boundary value problems [4]. The existence and uniqueness results have been studied extensively in [3]. Twizell [4] provided finite difference solution to general sixth-order BVPs. Other numerical methods emplyed for the solution of sixth order BVPs include but not limited to; homotopy perturbation methods [5,1], Modified Decomposition Method [6], Adomian Decomposition Method with Greens function [7], Variational approach and Sinc-Galerkin methods [8], Chebyshev Collocation-path [9]. Recently, the Cubic B-Spline method [10], was applied to solution of sixth-order BVPs.

Linear Multistep Method (LMM) for the direct solution of higher order ordinary differential equation using Chebyshev series approximate solution with interpolation and collocation approach to derived continuous (LMM) have been of interest in recent times, see [11-15]. Continuous (LMM) have greater advantages over the discrete method in that they give better error estimate, provide a simplified form of coefficient for further analytical work at different points and guarantee easy approximation of solution at all interior points of the integration interval, [12]. Continuous linear multistep method (LMM) is used via block technique, to formulate Finite Difference Methods (FDMs) using polynomials as basis functions, thus using multistep interpolation and collocation, continuous FDMs are derived which are assembled and solved simultaneously to obtain approximations $y_i \approx y(x_i)$, for $i = 1, \dots, N-1$ to the solution of (1) at points $x_i i = 1, \dots, N-1$. Another main advantage of this technique among others metioned earlier is that, the block method to be derived is self starting. This means that it does not reqire any starting value from any external method or guess value.

2 Derivation of the Methods

The exact solution $y(x)$ of (1) is approximated by form

$$p(x) = y(x) \simeq \sum_{i=0}^{r+s-1} \rho_i T_i(x) \tag{3}$$

with the sixth derivative given by as

$$p^{(vi)}(x) = y^{(vi)}(x) \simeq \sum_{i=0}^{r+s-1} \rho_i T_i^{(vi)}(x) \tag{4}$$

Where $x \in [a, b]$, ρ_i's are coefficient to be determined, $T_i(x)$ are Chebyshev polynomial of degree $r + s -$

1, r is the number of interpolation points that satisfies $6 < r \leq k$, s is the number of collocation points satisfying $0 < s \leq k+1$, and k is the step number. Here the following conditions are imposed;

$$\begin{cases} Y(x_{n+j}) = y_{n+j}, \quad j = 0,1,\dots,r-1 \\ Y^{(vi)}(x_{n+j}) = f_{n+j}, \quad j = 0,1,\dots,s-1, \quad \text{where} \quad f_{n+j} = y_{n+j}^{(vi)} \end{cases} \tag{5}$$

Here, $y_{n+j} = y(x_{n+j})$, $f_{n+j} = f(x_{n+j}, y_{n+j})$. Thus, we define the k-step linear multistep method as

$$Y(x) = \sum_{i=0}^{r-1} \alpha_i(t) y_{n+i} + h^6 \sum_{i=0}^{s-1} \beta_i(t) f_{n+i} \tag{6}$$

with the following derivatives

$$\begin{cases} Y'(x) = \frac{1}{h}\left(\sum_{i=0}^{r-1} \alpha'_i(x) y_{n+i} + h^6 \sum_{i=0}^{s-1} \beta'_i(x) f_{n+i}\right) \\ \vdots \\ Y^{(v)}(x) = \frac{1}{h^5}\left(\sum_{i=0}^{r-1} \alpha_i^{(5)}(x) y_{n+i} + h^6 \sum_{i=0}^{s-1} \beta_i^{(5)}(x) f_{n+i}\right) \end{cases} \tag{7}$$

where $\alpha_i(t)$ and $\beta_i(t)$ are continuous coefficients having derivatives $\alpha_i^j(t)$ and $\beta_i^j(t)$, $j = 1(1)5$. Let the solution of (1) be sought on the partition

π_N: $a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots < x_N = b$

of the integration interval $[a, b]$ with a constant step size h, given by $h = \frac{b-a}{N}$:
Interpolating (3) at x_{n+i}; $i = 0,1,2,\dots,r-1$ and collocating (4) at x_{n+s}; $s = 0,1,2,\dots,s-1$ leads to the following systems of equations:

$$\sum_{i=0}^{r+s-1} \rho_i T_i(x) = y_{n+i} \tag{8}$$

$$\sum_{i=0}^{r+s-1} \rho_i T_i^{(\mu)}(x) = f_{n+i} \tag{9}$$

where $\rho_i's$ are coefficients of the Chebyshev polynomials. Thus we define the following interpolation and collocation in a single matrix as follows

$$A = \begin{pmatrix} T_0(x_n) & T_1(x_n) & \cdot & \cdot & \cdot & T_{r+s-1}(x_n) \\ T_0(x_{n+1}) & T_1(x_{n+1}) & \cdot & \cdot & \cdot & T_{r+s-1}(x_{n+1}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ T_0(x_{n+r-1}) & T_1(x_{n+r-1}) & \cdot & \cdot & \cdot & T_{r+s-1}(x_{n+r-1}) \\ T_0^{(vi)}(x_n) & T_1^{(vi)}(x_n) & \cdot & \cdot & \cdot & T_{r+s-1}^{(vi)}(x_n) \\ T_0^{(vi)}(x_{n+1}) & T_1^{(vi)}(x_{n+1}) & \cdot & \cdot & \cdot & T_{r+s-1}^{(vi)}(x_{n+1}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ T_0^{(vi)}(x_{n+s-1}) & T_1^{(vi)}(x_{n+s-1}) & \cdot & \cdot & \cdot & T_{r+s-1}^{(vi)}(x_{n+s-1}) \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} \rho_0 & \rho_1 & \cdot & \cdot & \cdot & \rho_{r+s-1} \end{pmatrix}^T$$

$$\underline{c} = \begin{pmatrix} y_n & y_{n+1} & \cdot & \cdot & \cdot & y_{n+r-1} & f_n & f_n & \cdot & \cdot & \cdot & f_{n+s-1} \end{pmatrix}^T$$

So that

$$A\underline{b} = \underline{c} \qquad (10)$$

$$\underline{w} = \begin{pmatrix} T_0(x) & T_1(x) & \cdot & \cdot & \cdot & T_{r+s-1}(x) \end{pmatrix}^T$$

Hence we state the following theorem

Theorem 1 [14]. *Let (5) be satisfied, the continuous k-step LMM (6), and (7) are respectively derived from the equation*

$$Y(x) = \underline{c}^T (A^{-1})^T \underline{w} \qquad (11)$$

Proof. Given the continuous scheme

$$Y(x) = \sum_{i=0}^{r+s-1} \rho_i T_i(x) \qquad (12)$$

where $x \in [a, b]$, ρ_i's are unknown coefficients, $T_i(x)$'s are the Chebyshev polynomial basis functions of degree $r + s - 1$. We can clearly write (12) as

$$Y(x) = \rho_0 T_0(x) + \rho_1 T_1(x) + \rho_2 T_2(x) + \cdots + \rho_{r+s-2} T_{r+s-2}(x) + \rho_{r+s-1} T_{r+s-1}(x) \qquad (13)$$

Then (13) can be written compactly in vector form as

$$Y(x) = \underline{b}^T \underline{w} \qquad (14)$$

From (10), by left inverse cancelation law we have

$$\underline{b} = A^{-1} \underline{c} \qquad (15)$$

Hence, by (15) we have

$$Y(x) = \underline{c}^T (A^{-1})^T \underline{w} \qquad (16)$$

as required.

It is note worthy that the continuous methods (6) is equivalent to (16) and will be used to produce the main and additional methods which gives a total of 36 equations and are combined to provide all approximations on the entire interval for the solution of (1).

2.1 Specification

Now, applying Theorem (1), with $k = 6$, $r = 6$, $s = 7$, the six-step linear multistep method is of the form

$$Y(x) = \sum_{i=0}^{5} \alpha_i y_{n+i} + h^6 \sum_{i=0}^{6} \beta_i f_{n+i} \qquad (17)$$

with the following derivatives

$$Y^{(j)}(x) = \frac{1}{h^j} \left(\sum_{i=0}^{5} \alpha_i y_{n+i} + h^6 \sum_{i=0}^{6} \beta_i f_{n+i} \right), \quad j = 1(1)5 \qquad (18)$$

Evaluating (17) and (18) at the point $x = x_{n+6}$, the coefficients of the main methods are as shown in Table 1.

Table 1. Coefficients of main formulae for $y_{n+6}^{(i)}, \quad i = 0(1)5$

	α_0	α_1	α_2	α_3	α_4	α_5	β_0	β_1	β_2	β_3	β_4	β_5	β_6
y_{n+6}	-1	6	-15	20	-15	6	$\frac{1}{30240}$	$\frac{41}{5040}$	$\frac{2189}{10080}$	$\frac{4153}{7560}$	$\frac{2189}{10080}$	$\frac{41}{5040}$	$\frac{1}{30240}$
hy'_{n+6}	$\frac{-137}{60}$	$\frac{27}{2}$	33	$\frac{127}{3}$	$\frac{-117}{4}$	$\frac{87}{10}$	$\frac{809}{6652800}$	$\frac{20249}{1108800}$	$\frac{122809}{246400}$	$\frac{2155337}{1663200}$	$\frac{1322381}{2217600}$	$\frac{15203}{369600}$	$\frac{-133}{950400}$
$h^2y''_{n+6}$	$\frac{-15}{4}$	$\frac{65}{3}$	$\frac{-307}{6}$	62	$\frac{-461}{12}$	$\frac{29}{3}$	$\frac{20479}{59875200}$	$\frac{289127}{9979200}$	$\frac{16484849}{19958400}$	$\frac{33409303}{14968800}$	$\frac{25205759}{19958400}$	$\frac{1606223}{9979200}$	$\frac{59}{171072[illegible]}$
$h^3y'''_{n+6}$	$\frac{-17}{4}$	$\frac{95}{4}$	$\frac{-107}{2}$	$\frac{121}{2}$	$\frac{-137}{4}$	$\frac{31}{4}$	$\frac{73}{241920}$	$\frac{4037}{120960}$	$\frac{227791}{241920}$	$\frac{54581}{20160}$	$\frac{471839}{241920}$	$\frac{58501}{120960}$	$\frac{287}{34560}$
$h^4y_{n+6}^{(iv)}$	-3	16	-34	36	-19	4	$\frac{-4703}{1814400}$	$\frac{4411}{100800}$	$\frac{24517}{40320}$	$\frac{39817}{18144}$	$\frac{8319}{4480}$	$\frac{35857}{33600}$	$\frac{16981}{259200}$
$h^5y_{n+6}^{(v)}$	-1	5	-10	10	-5	1	$\frac{-457}{40320}$	$\frac{2741}{30240}$	$\frac{-4267}{120960}$	$\frac{18841}{15120}$	$\frac{53863}{120960}$	$\frac{44161}{30240}$	$\frac{5257}{17280}$

The additional methods are obtained by evaluating (18) at the point $x = x_j, j = 0(1)5$. The coefficients of the additional methods are as shown in Tables 2-6.

Table 2. Coefficients of main formulae for $y'_j, \ j = 0(1)5$

	α_0	α_1	α_2	α_3	α_4	α_5	β_0	β_1	β_2	β_3	β_4	β_5	β_6
hy'_0	$\frac{-137}{60}$	5	-5	$\frac{10}{3}$	$\frac{-5}{4}$	$\frac{1}{5}$	$\frac{43}{199584}$	$\frac{-67}{2970}$	$\frac{-4177}{41580}$	$\frac{-2593}{62370}$	$\frac{-851}{332640}$	$\frac{13}{41580}$	$\frac{-23}{498960}$
hy'_1	$\frac{-1}{5}$	$\frac{-13}{12}$	2	-1	$\frac{1}{3}$	$\frac{-1}{20}$	$\frac{-197}{4989600}$	$\frac{3601}{1663200}$	$\frac{33107}{1663200}$	$\frac{3929}{356400}$	$\frac{37}{151200}$	$\frac{61}{1663200}$	$\frac{-17}{4989600}$
hy'_2	$\frac{1}{20}$	$\frac{-1}{2}$	$\frac{-1}{3}$	1	$\frac{-1}{4}$	$\frac{1}{30}$	$\frac{-101}{19958400}$	$\frac{-281}{665280}$	$\frac{-58309}{6652800}$	$\frac{-37133}{4989600}$	$\frac{311}{6652800}$	$\frac{43}{475200}$	$\frac{223}{19958400}$
hy'_3	$\frac{-1}{30}$	$\frac{1}{4}$	-1	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{-1}{20}$	$\frac{7}{570240}$	$\frac{25}{133056}$	$\frac{46909}{6652800}$	$\frac{4153}{453600}$	$\frac{1249}{6652800}$	$\frac{277}{3326400}$	$\frac{-223}{19958400}$
hy'_4	$\frac{1}{20}$	$\frac{-1}{3}$	1	-2	$\frac{13}{12}$	$\frac{1}{5}$	$\frac{-2}{155925}$	$\frac{-263}{831600}$	$\frac{-2591}{237600}$	$\frac{-24979}{1247400}$	$\frac{-1741}{831600}$	$\frac{13}{831600}$	$\frac{17}{4989600}$
hy'_5	$\frac{-1}{5}$	$\frac{5}{4}$	$\frac{-10}{3}$	5	-5	$\frac{137}{60}$	$\frac{2}{99792}$	$\frac{529}{332640}$	$\frac{653}{15120}$	$\frac{49319}{498960}$	$\frac{559}{23760}$	$\frac{-179}{332640}$	$\frac{23}{498960}$

Table 3. Coefficients of main formulae for $y''_j, \quad j = 0(1)5$

	α_0	α_1	α_2	α_3	α_4	α_5	β_0	β_1	β_2	β_3	β_4	β_5	β_6
$h^2y''_0$	$\frac{15}{4}$	$\frac{-77}{6}$	$\frac{107}{6}$	-13	$\frac{67}{12}$	$\frac{-5}{6}$	$\frac{-67}{748440}$	$\frac{650899}{4989600}$	$\frac{4476217}{9979200}$	$\frac{1286639}{7484400}$	$\frac{57881}{4989600}$	$\frac{-7649}{4989600}$	$\frac{6527}{2993760}$
$h^2y''_1$	$\frac{5}{6}$	$\frac{-5}{4}$	$\frac{-1}{3}$	$\frac{7}{6}$	$\frac{-1}{2}$	$\frac{1}{12}$	$\frac{2851}{14968800}$	$\frac{-83773}{9979200}$	$\frac{-229219}{4989600}$	$\frac{-252347}{14968800}$	$\frac{-7129}{4989600}$	$\frac{2423}{9979200}$	$\frac{-103}{2993760}$
$h^2y''_2$	$\frac{-1}{12}$	$\frac{4}{3}$	$\frac{-5}{2}$	$\frac{4}{3}$	$\frac{-1}{12}$	0	$\frac{-379}{11975040}$	$\frac{11069}{9979200}$	$\frac{179219}{19958400}$	$\frac{15613}{14968800}$	$\frac{689}{19958400}$	$\frac{-283}{9979200}$	$\frac{283}{5987520}$
$h^2y''_3$	0	$\frac{-1}{12}$	$\frac{4}{3}$	$\frac{-5}{2}$	$\frac{4}{3}$	$\frac{-1}{12}$	$\frac{283}{59875200}$	$\frac{-323}{4989600}$	$\frac{24119}{19958400}$	$\frac{65969}{7484400}$	$\frac{24119}{19958400}$	$\frac{-323}{4989600}$	$\frac{283}{5987520}$
$h^2y''_4$	$\frac{1}{12}$	$\frac{-1}{2}$	$\frac{7}{6}$	$\frac{-1}{3}$	$\frac{-5}{4}$	$\frac{5}{6}$	$\frac{59}{29937600}$	$\frac{-881}{1247400}$	$\frac{-22531}{1247400}$	$\frac{-20926}{467775}$	$\frac{-90983}{9979200}$	$\frac{269}{623700}$	$\frac{-103}{2993760}$
$h^2y''_5$	$\frac{-5}{6}$	$\frac{61}{12}$	-13	$\frac{107}{6}$	$\frac{-77}{6}$	$\frac{15}{4}$	$\frac{-41}{5987520}$	$\frac{70073}{9979200}$	$\frac{1791667}{9979200}$	$\frac{6600103}{14968800}$	$\frac{1347487}{9979200}$	$\frac{-16123}{9979200}$	$\frac{6527}{2993760}$

Table 4. Coefficients of main formulae for $y'''_j, \quad j = 0(1)5$

	α_0	α_1	α_2	α_3	α_4	α_5	β_0	β_1	β_2	β_3	β_4	β_5	β_6
$h^3y'''_0$	$\frac{-17}{4}$	$\frac{71}{4}$	$\frac{-59}{2}$	$\frac{49}{2}$	$\frac{-41}{4}$	$\frac{7}{4}$	$\frac{-395}{48384}$	$\frac{-54319}{120960}$	$\frac{-248561}{241920}$	$\frac{-22541}{60480}$	$\frac{-4513}{241920}$	$\frac{29}{24192}$	$\frac{-13}{80640}$
$h^3y'''_1$	$\frac{-7}{4}$	$\frac{25}{4}$	$\frac{-17}{2}$	$\frac{11}{2}$	$\frac{-7}{4}$	$\frac{1}{4}$	$\frac{-5}{48384}$	$\frac{871}{120960}$	$\frac{-17519}{241920}$	$\frac{-1219}{20160}$	$\frac{409}{241920}$	$\frac{-25}{24192}$	$\frac{31}{241920}$
$h^3y'''_2$	$\frac{-1}{4}$	$\frac{-1}{4}$	$\frac{5}{2}$	$\frac{-7}{2}$	$\frac{7}{4}$	$\frac{-1}{4}$	$\frac{11}{80640}$	$\frac{25}{24192}$	$\frac{15527}{241920}$	$\frac{731}{12096}$	$\frac{-409}{241920}$	$\frac{25}{24192}$	$\frac{-31}{241920}$
$h^3y'''_3$	$\frac{1}{4}$	$\frac{-7}{4}$	$\frac{7}{2}$	$\frac{-5}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{-11}{80640}$	$\frac{-121}{120960}$	$\frac{-2707}{48384}$	$\frac{-4153}{60480}$	$\frac{401}{241920}$	$\frac{-25}{24192}$	$\frac{31}{241920}$
$h^3y'''_4$	$\frac{-1}{4}$	$\frac{7}{4}$	$\frac{-11}{2}$	$\frac{17}{2}$	$\frac{-25}{4}$	$\frac{7}{4}$	$\frac{11}{80640}$	$\frac{121}{120960}$	$\frac{13543}{241920}$	$\frac{4651}{60480}$	$\frac{-2393}{241920}$	$\frac{121}{120960}$	$\frac{-31}{241920}$
$h^3y'''_5$	$\frac{-7}{4}$	$\frac{41}{4}$	$\frac{-49}{2}$	$\frac{59}{2}$	$\frac{-71}{4}$	$\frac{17}{4}$	$\frac{-17}{241920}$	$\frac{184}{120960}$	$\frac{91529}{241920}$	$\frac{61799}{60480}$	$\frac{109457}{241920}$	$\frac{851}{120960}$	$\frac{13}{80640}$

Table 5. Coefficients of main formulae for $y_j^{(iv)}$, $j = 0(1)5$

	α_0	α_1	α_2	α_3	α_4	α_5	β_0	β_1	β_2	β_3	β_4	β_5	β_6
$h^4y_0^{(iv)}$	3	-14	26	-24	11	-2	$\frac{118687}{1814400}$	$\frac{11679}{11200}$	$\frac{16201}{13440}$	$\frac{49577}{90720}$	$\frac{-1751}{40320}$	$\frac{1951}{100800}$	$\frac{-4883}{1814400}$
$h^4y_1^{(iv)}$	2	-9	16	-14	6	-1	$\frac{-5003}{1814400}$	$\frac{10279}{151200}$	$\frac{66761}{120960}$	$\frac{1823}{9072}$	$\frac{2173}{120960}$	$\frac{-481}{151200}$	$\frac{817}{1814400}$
$h^4y_2^{(iv)}$	1	-4	6	-4	1	0	$\frac{757}{1814400}$	$\frac{-4247}{302400}$	$\frac{-16901}{120960}$	$\frac{-239}{18144}$	$\frac{-11}{24192}$	$\frac{113}{302400}$	$\frac{-113}{1814400}$
$h^4y_3^{(iv)}$	0	1	-4	6	-4	1	$\frac{-113}{1814400}$	$\frac{43}{50400}$	$\frac{-619}{40320}$	$\frac{-6239}{45360}$	$\frac{-619}{40320}$	$\frac{43}{50400}$	$\frac{-113}{1814400}$
$h^4y_4^{(iv)}$	-1	6	-14	16	-9	2	$\frac{-53}{1814400}$	$\frac{2573}{302400}$	$\frac{26213}{120960}$	$\frac{48641}{90720}$	$\frac{9367}{120960}$	$\frac{-1787}{302400}$	$\frac{817}{1814400}$
$h^4y_5^{(iv)}$	-2	11	-24	26	-14	3	$\frac{937}{1814400}$	$\frac{1979}{151200}$	$\frac{54709}{120960}$	$\frac{58951}{45360}$	$\frac{119297}{120960}$	$\frac{12739}{151200}$	$\frac{-4883}{1814400}$

Table 6. Coefficients of main formulae for $y_j^{(v)}$, $j = 0(1)5$

	α_0	α_1	α_2	α_3	α_4	α_5	β_0	β_1	β_2	β_3	β_4	β_5	β_6
$h^5y_0^{(v)}$	-1	5	-10	10	-5	1	$\frac{-2453}{8064}$	$\frac{-8783}{6048}$	$\frac{-5519}{24192}$	$\frac{-301}{432}$	$\frac{6107}{24192}$	$\frac{-499}{6048}$	$\frac{275}{24192}$
$h^5y_1^{(v)}$	-1	5	-10	10	-5	1	$\frac{197}{17280}$	$\frac{-11359}{30240}$	$\frac{-120517}{120960}$	$\frac{-1159}{15120}$	$\frac{-9887}{120960}$	$\frac{661}{30240}$	$\frac{-13}{4480}$
$h^5y_2^{(v)}$	-1	5	-10	10	-5	1	$\frac{-347}{120960}$	$\frac{241}{6048}$	$\frac{-26539}{120960}$	$\frac{-1741}{5040}$	$\frac{673}{17280}$	$\frac{-383}{30240}$	$\frac{191}{120960}$
$h^5y_3^{(v)}$	-1	5	-10	10	-5	1	$\frac{13}{8064}$	$\frac{-5}{864}$	$\frac{35099}{120960}$	$\frac{4153}{15120}$	$\frac{-8831}{120960}$	$\frac{421}{30240}$	$\frac{-191}{120960}$
$h^5y_4^{(v)}$	-1	5	-10	10	-5	1	$\frac{-187}{120960}$	$\frac{629}{30240}$	$\frac{21557}{120960}$	$\frac{13529}{15120}$	$\frac{52807}{120960}$	$\frac{-137}{4320}$	$\frac{13}{4480}$
$h^5y_5^{(v)}$	-1	5	-10	10	-5	1	$\frac{71}{24192}$	$\frac{-83}{6048}$	$\frac{1033}{3456}$	$\frac{631}{1008}$	$\frac{29357}{24192}$	$\frac{2321}{6048}$	$\frac{-275}{24192}$

The formulae in (17) and (18) together form the block method we refer to as the Boundary Value Method (BVM) applied for solution of BVPs, the formulae in (17) and (18) for $n = 0(6)N - 6$ are considered at the same time where N is the number of subintervals of the interval $[a, b]$.

2.2 Analysis of the methods

In this section, the local truncation error, order, consistency and convergence of theBVM are discussed.

2.2.1 Local truncation error and order

The methods derived in (17) and (18) are associated with the linear differential operator $\mathcal{L}[y(x);\ h]$ defined by

$$\mathcal{L}[y(x);\ h] = y(x + jh) - \sum_{i=0}^{5} h^i \alpha y(x + jh) - h^6 \sum_{i=0}^{6} \beta f(x + jh) \tag{19}$$

and

$$\mathcal{L}[y(x);\ h] = y^{(l)}(x + jh) - \frac{1}{h^j}\left(\sum_{i=0}^{5} \alpha y(x + jh) + h^6 \sum_{i=0}^{6} \beta f(x + jh)\right) \tag{20}$$

Expanding (19) (also (20)) in Taylor series, we obtain the following linear combination of the constants C_i's of the form

$$\mathcal{L}[y(x);\ h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^q y^{(p)}(x) + O(h^{(p+1)}) \tag{21}$$

So that, the LMM (17) is of order p if

$$C_0 = C_1 = C_2 = \cdots = C_{p+5} = 0, \quad \text{and} \quad C_{p+6} \neq 0$$

in which

$$\mathcal{L}[y(x);\ h] = C_{p+6}h^{p+6}y^{(p+6)}(x) + O(h^{p+7}) \tag{22}$$

In this case, C_{p+6} is the principal error constant, see [16]. The local truncation error associated with the main methods (17);

$v_0 = -\frac{y^{(14)}(x)h^{14}}{57600} + O(h^{15}$, $hv'_0 = \frac{2687y^{(13)}(x)h^{13}}{60540480} + O(h^{14})$, $h^2v''_0 = -\frac{179y^{(13)}(x)h^{13}}{950400} + O(h^{14})$, $h^3v'''_0 = \frac{97y^{(13)}(x)h^{13}}{997920} + O(h^{14})$, $h^4v_0^{(iv)} = \frac{199y^{(13)}(x)h^{13}}{86400} + O(h^{14})$, $h^5v_0^{(v)} = -\frac{3389y^{(13)}(x)h^{13}}{362880} + O(h^{14})$.

The local truncation error associated with the additional methods (18) are are obtained in the same way. Thus, the methods (17) and (18) are consistent (with $p > 1$).

2.3 Convergence analysis

The convergence of the method is investigated using the a Toeplitz seven-band matrix. Consider the system

$$\begin{cases} \text{i.} \quad A_{N-k_1}Y = h^6B_{N-k_1}F + \tau, \\ \text{ii.} \quad A_{N-k_1}e = h^6B_{N-k_1}\delta F + \tau + g, \end{cases} \tag{23}$$

where

$$e = \left[e_{k_1}, \dots, e_{N-1}\right]^T, \quad e_i = y(t_i) - y_i,$$

$\delta F = \left[f(t_{k_1}, y(t_{k_1}), y'(t_{k_1}), \cdots, y^{(5)}(t_{k_1})) - f_{k_1}, \dots, f(t_{N-1}, y(t_{N-1}), y'(t_{N-1}), \cdots, y^{(5)}(t_{N-1})) - \right.$ fN−1T,

$$F = [f(x_{k_1}, y(x_{k_1}), y'(x_{k_1}), \cdots, y^{(5)}(x_{k_1})), \dots, f(x_{N-1}, y(x_{N-1}), y'(x_{N-1}), \cdots, y^{(5)}(x_{N-1}))]^T,$$

$Y = [y_{k_1}, \dots, y_{N-1}]^T$ are $(N - k_1) \times 1$vectors,

$$A_{N-k_1} = \begin{bmatrix} \alpha_{k_1} & \cdots & \alpha_k & & & \\ \vdots & & & \ddots & & \\ \alpha_0 & & \ddots & & \ddots & \\ & \ddots & & & & \ddots \\ & & \ddots & & \ddots & & \alpha_{k_1} \\ & & & \ddots & & & \vdots \\ & & & & \alpha_0 & \cdots & \alpha_{k_1} \end{bmatrix},$$

$$B_{N-k_1} = \begin{bmatrix} \beta_{k_1} & \cdots & \beta_k & & & \\ \vdots & & & \ddots & & \\ \beta_0 & & \ddots & & \ddots & \\ & \ddots & & & & \ddots \\ & & \ddots & & \ddots & & \beta_{k_1} \\ & & & \ddots & & & \vdots \\ & & & & \beta_0 & \cdots & \beta_{k_1} \end{bmatrix} \tag{24}$$

are all $(N - k_1) \times (N - k_1)$ matrices. τ is the vector containing all local errors while g_e contains the errors at the boundary.

$$g_e = -\begin{bmatrix} \sum_{i=0}^{k_1-1} (\alpha_i e_i - h^6 \beta_i \delta F_i) \\ \sum_{i=0}^{k_1-2} (\alpha_i e_{i+1} - h^6 \beta_i \delta F_{i+1}) \\ \vdots \\ \sum_{i=0}^{k_1-k_1} \left(\alpha_i e_{k_1-1} - h^6 \beta_i \delta F_{k_1-1}\right) \\ \vdots \\ \sum_{i=1}^{k_2-1} \left(\alpha_{i+k_1+1} e_{N-k_1+i-1} - h^6 \beta_{i+k_1+1} \delta F_{N-k_1+i-1}\right) \\ \sum_{i=1}^{k_2} \left(\alpha_{i+k_1} e_{N-k_1+i-1} - h^6 \beta_{i+k_1} \delta F_{N-k_1+i-1}\right) \end{bmatrix} \tag{25}$$

By the Boundary value method [11], fix the values $k = 3$, $k_1 = 3$. Specifically,

$$A_{N-3} = \begin{bmatrix} 20 & -15 & 6 & -1 & & & & & \\ -15 & 20 & -15 & 6 & -1 & & & & \\ 6 & 15 & 20 & -15 & 6 & -1 & & & \\ -1 & 6 & -15 & 20 & -15 & 6 & -1 & & \\ & & & \cdots & \cdots & \cdots & & & \\ & & -1 & 6 & -15 & 20 & -15 & 6 & -1 \\ & & & -1 & 6 & -15 & 20 & -15 & 6 \\ & & & & -1 & 6 & -15 & 20 & 15 \\ & & & & & -1 & 6 & -15 & 20 \end{bmatrix} \tag{26}$$

$$B_{N-3} = -\frac{1}{30240}\begin{bmatrix} 16612 & 6567 & 246 & 1 & & & & & \\ 6567 & 16612 & 6567 & 246 & 1 & & & & \\ 246 & 6567 & 16612 & 6567 & 246 & 1 & & & \\ 1 & 246 & 6567 & 16612 & 6567 & 246 & 1 & & \\ & & & \cdots & \cdots & \cdots & & & \\ & & 1 & 246 & 6567 & 16612 & 6567 & 246 & 1 \\ & & & 1 & 246 & 6567 & 16612 & 6567 & 246 \\ & & & & 1 & 246 & 6567 & 16612 & 6567 \\ & & & & & 1 & 246 & 6567 & 16612 \end{bmatrix} \tag{27}$$

$$g_e = \begin{bmatrix} -\frac{1}{30240} h^6 \delta f_0 + e_0 - \frac{41}{5040} h^6 \delta f_1 - 6\, e_1 - \frac{2189}{10080} h^6 \delta f_2 + 15\, e_2 \\ -\frac{1}{30240} h^6 \delta f_1 + e_1 - \frac{41}{5040} h^6 \delta f_2 - 6\, e_2 \\ -\frac{1}{30240} h^6 \delta f_2 + e_2 \\ \vdots \\ -\frac{1}{30240} h^6 \delta f_N + e_N \\ -\frac{41}{5040} h^6 \delta f_N - 6\, e_N - \frac{1}{30240} h^6 \delta f_{N+1} + e_{N+1} \\ -\frac{2189}{10080} h^6 \delta f_N + 15\, e_N - \frac{41}{5040} h^6 \delta f_{N+1} - 6\, e_{N+1} - \frac{1}{30240} h^6 \delta f_{N+2} + e_{N+2} \end{bmatrix} \tag{28}$$

2.3.1 Determinant of A_{N-3}

Following Hoskins and Ponzo [9], the determinant of A_{N-3} is given by the following theorem

Lemma 2 [17]. *If $N \geq 3$, then determinant of the sixth order difference operator matrix A_{N-3} is given by*

$$\det\ A_{N-3} = \frac{(N-2)\,(N-1)^2 N^3\,(N+1)^2\,(N+2)}{8640}$$

2.3.2 The inverse of A_{N-3}

Lemma 3 [17]. Let A_{N-3} be an even order difference operator matrix (26) then from [17], the elements on the diagonal and below are given by

$$\begin{aligned}\alpha_{ij} &= -\frac{(N-2-i)(N-1-i)(N-i)}{240(N-2)(N-1)N(N+1)(N+2)}[(i+1)(i+2)(j+2)(j+1)j(j-1)(j-2)(N+1)(N+2)\\ &- 2i(i+2)(j+3)(j+2)(j+1)j(j-1)(N-2)(N+2)\\ &+i(i+1)(j+4)(j+3)(j+2)(j+1)j(N-1)(N-2)], \quad i \geq j\end{aligned} \tag{29}$$

Since A_{N-3}^{-1} is a symmetric matrix, hence on interchanging i and j in the previous equation, the terms for α_{ij} for $i \leq j$ can be obtained. The results obtained so far are summarized in the next lemma.

Lemma 4 [17]. *The symmetric matrix A_{N-3} is irreducible and monotone and if $A_{N-3}^{-1} = [\alpha_{ij}]$, then $A_{N-k_1}^{-1}$ is symmetric, satisfying $A^{-1} > 0$ where*

$$\begin{aligned}\alpha_{ij} &= -\frac{(N-2-j)(N-1-j)(N-j)}{240(N-2)(N-1)N(N+1)(N+2)}[(j+1)(j+2)(i+2)(i+1)i(i-1)(i-2)(N+1)(N+2)\\ &- 2j(j+2)(i+3)(i+2)(i+1)i(i-1)(N-2)(N+2)\\ &+j(j+1)(i+4)(i+3)(i+2)(i+1)i(N-1)(N-2)], \quad i \leq j\end{aligned} \tag{30}$$

In particular, for $N = 10$

$$A_{N-3}^{-1} = \frac{1}{660}\begin{bmatrix} 252 & 504 & 630 & 600 & 450 & 252 & 84 \\ 504 & 1232 & 1680 & 1680 & 1300 & 744 & 252 \\ 630 & 1680 & 2555 & 2730 & 2205 & 1300 & 450 \\ 600 & 1680 & 2730 & 3180 & 2730 & 1680 & 600 \\ 450 & 1300 & 2205 & 2730 & 2555 & 1680 & 630 \\ 252 & 744 & 1300 & 1680 & 1680 & 1232 & 504 \\ 84 & 252 & 450 & 600 & 630 & 504 & 252 \end{bmatrix} \tag{31}$$

2.3.3 The infinity-norm of A_{N-3}^{-1}

The inverse can be used in verifying the formulas (32) and (33) which will be established next. Define $R_i = \sum_{j=1}^{N-3} \alpha_{ij}$, then

$$\| A^{-1} \|_\infty = \max_i |R_i|.$$

It follows that

$$R_i = \frac{1}{720}(i+1)(i+2)(i-N)(i-N+2)(i-N+1)i \tag{32}$$

This gives

$$\begin{aligned}\frac{dR_i}{di} &= \frac{1}{720}(2\,i-N+2)[3\,i^4 - 6\,i^3(N-3) + 3\,i^2(N-3)^2\\ &-6\,i^3 + 6\,i(N-3)^2 - 13\,i^2 + 22\,i(N-3) + 2\,(N-3)^2 + 16\,i + 10\,(N-3) + 12] = 0\end{aligned} \tag{33}$$

Consider R_i as a function of real variable i. Then R_i is symmetric in the interval $[1, N-3]$ and it can be easily shown that R_i has its maximum for $i = (N-2)/2$ for even values of N while R_i has its maximum for $i = (N-3)/2$ for odd N. Now $d^2R_i/di^2 < 0$ for this value of i since A_{N-3} is a positive definite matrix. The infinity norm of A_{N-3}^{-1} must be bounded to an integral value of i. Thus,

$$\| A_{N-3}^{-1} \|_\infty \leq R_{(N-2)/2}.$$

Substituting $i = (N-2)/2$ in (32) to find $R_{(N-2)/2}$. This gives

$$\| A_{N-3}^{-1} \|_\infty \leq R_{(N-2)/2} \begin{array}{l} = \frac{1}{46080}(N+2)^2N^2(N-2)^2, \quad forevenN \\ = \frac{1}{46080}(N-3)(N-1)^2(N+1)^2(N+3)foroddN \end{array} \tag{34}$$

Lemma 5 [17]. *The infinity norm of* A_{N-3}^{-1} *is given by*

$$\| A_{N-3}^{-1} \|_\infty \leq R_{(N-2)/2} = \frac{1}{\infty} \leq R_{(N-2)/2} \begin{array}{l} = \frac{1}{46080}h^{-6}(b-a-5h)(b-a-3h)^2(b-a-h)^2(b-a+h) \\ = O(h^{-6}) \end{array} \tag{35}$$

for odd N.

Proof. The proof follows easily, using

$$t_i = a + ih, \quad i = 0,1,\cdots,N+2, \quad t_0 = a, \qquad t_{N+2} = b, \quad h = (b-a)/(N+2)$$

and (35)

Lemma 6 [17]. *The matrix* A_{N-3} *is nonsingular, provided that* $LP \geq 0$ *where*

$$P = (1/46080)\, h^{-6}(b-a-5h)(b-a-3h)^2(b-a-h)^2(b-a+h) = O(h^{-6}),$$

and L is the Lipschitz constant of f.

2.3.4 Error bound

The error equation is (23i.). Let L be the Lipschitz constant of f. Since for sufficiently large N, the matrix A_{N-3} is always nonsingular, then it can be shown that

$$\| e \|_\infty \leq \frac{\| A_{N-3}^{-1} \|_\infty \| \tau \|_\infty}{1 - h^6 \| A_{N-3}^{-1} \|_\infty L \| B_{N-3} \|},$$

where $\| \tau \|_\infty = \frac{3389}{362880}h^{13}M_{11}$, $\quad M_{11} = \max_x|y^{(13)}(x)|$, $\quad \| B_{N-3} \| = -1$,

Thus,

$$\begin{aligned} \| e \|_\infty \quad &\leq \frac{(3389\, M_{11}h^{-6} + 362880\, Gh^{-13} + 362880\, Eh^{-18})\, Ph^{13}}{362880(1+LP)} \\ &= Jh^{13}, \end{aligned}$$

where the constant

$$J = \frac{1}{362880(1+LP)}(3389\, M_{11}Ph^{-6} + 362880\, PGh^{-13} + 362880\, PEh^{-18})$$

The summarization of the details above are presented in the next theorem.

Theorem 7 [17]. *Let* $y(x)$ *be the exact solution of the continuous boundary value problem (1.1) and let* y_i, $i = 0,1,\ldots,N-3$, *satisfy the discrete boundary value problem (1i). Further, if* $e_i = |y(t_i) - y_i|$, *then* $\| E \|_\infty = O(h^{13})$.

3 Numerical Examples

In this part, we implement our derived method using numerical examples to show the high level of accuracy and efficiency of this method.

Example 1.. Consider the following nonlinear BVP discussed in [18] of the form

$$\begin{aligned} &y^{(vi)}(x) = e^{-x}y^2(x). \quad x \in [0,1], \\ &y(0) = y''(0) = y^{(iv)}(0) = 1 \\ &y(1) = y''(1) = y^{(iv)}(1) = e \end{aligned} \tag{36}$$

which has an exact solution of $y(x) = e^x$.

Table 7. Comparison of maxium absolute error for different values of h

h	$1/8$	$1/16$	$1/32$	$1/64$
BVM	6.94×10^{-14}	3.34×10^{-16}	1.39×10^{-16}	4.25×10^{-17}
Khan and Khandelwal [18]	2.25×10^{-7}	2.19×10^{-8}	1.94×10^{-9}	1.35×10^{-9}

It can be seen that the BVM with different step sizes performs better than the method in [18]

Example 2. Consider the boundary value problem, discussed in [10,19].

$$\begin{aligned} &y^{(vi)} + xy = (-24 + 11x + x^3)e^x. \quad x \in [0,1] \\ &y(0) = y(1) = 0 \\ &y''(0) = 0, \quad y''(1) = -4e, \\ &y^{(iv)}(0) = -8, \quad y^{(iv)}(1) = -16e. \end{aligned} \tag{37}$$

The analytical solution of the above problem is $y(x) = x(1-x)e^x$.

Table 8. Absolute error with $h = 0.1$ obtained for example 2

x	Error for BVM	Error for Cubic B-Spline [10]
0.1	3.45×10^{-12}	3.81×10^{-5}
0.2	1.98×10^{-12}	1.59×10^{-4}
0.3	2.71×10^{-12}	3.41×10^{-4}
0.4	9.87×10^{-12}	5.33×10^{-4}
0.5	7.12×10^{-12}	6.74×10^{-4}
0.6	2.88×10^{-12}	7.08×10^{-4}
0.7	3.21×10^{-12}	6.08×10^{-4}
0.8	9.45×10^{-12}	3.91×10^{-4}
0.9	1.10×10^{-12}	1.35×10^{-4}

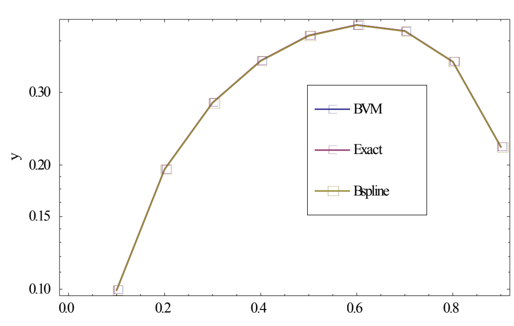

Fig. 1. Showing the exact solution in comparison with numerical solution

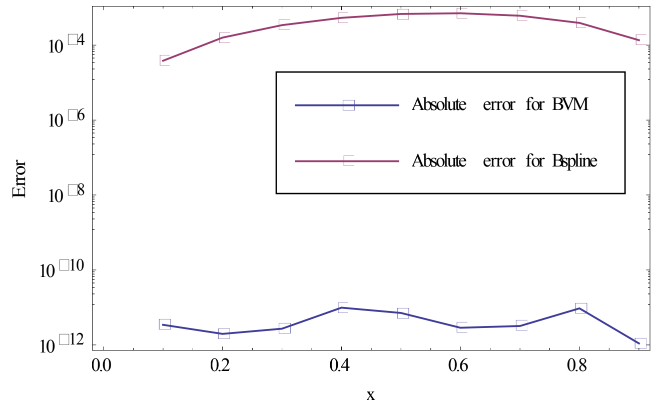

Fig. 2. Graph comparing the absolute errors for BVM and Cubic B-Spline in [10]

Table 9. Comparison of maximum absolute errors for example 2

h	$1/8$	$1/16$	$1/32$	$1/64$
BVM	1.25×10^{-11}	4.72×10^{-14}	9.66×10^{-17}	1.94×10^{-17}
Khan and Sultana [19]	2.25×10^{-7}	2.19×10^{-8}	1.94×10^{-9}	1.35×10^{-9}

Similarly as in example 1, the BVM with different step sizes performs better than the method in [19]

Fig. 1 shows that the numerical solutions compares favourably with the exact solution for h=0.1. Fig. 2 shows the comparison of absolute error obtained for example 2 with the BVM and the method in [10]. This evidently shows that the BVm performs better thatn those compared with.

Example 3. Consider the boundary value problem, discussed in [10,20]

$$\begin{cases} y^{(\mathrm{vi})}(x) - y(x) = -6\,e^{x} \\ y(0) = 1, \quad y(1) = 0, \\ y'(0) = 0, \quad y'(1) = -e, \\ y''(0) = -1, \quad y''(1) = -2e. \end{cases}$$

The analytical solution of the above problem is $y(x) = (1-x)e^{x}$.

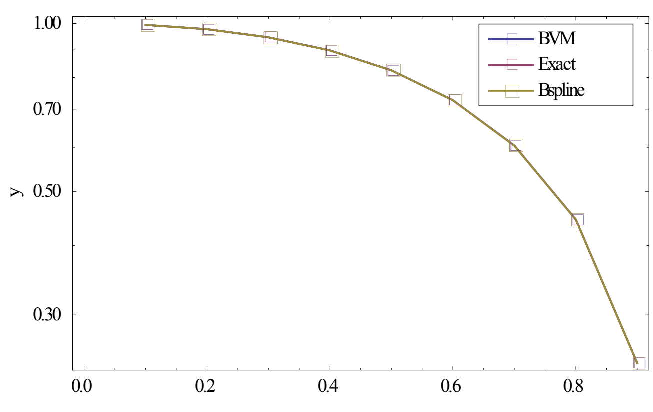

Fig. 3. Graph of exact and numerical solutions for $h = 0.1$

Table 10. Absolute error with $h = 0.1$ obtained for Example 3

x	Error for BVM $h = 0.1$.	Error for Cubic B-Spline [10]
0.1	1.23×10^{-15}	1.18×10^{-5}
0.2	1.01×10^{-15}	4.29×10^{-5}
0.3	1.27×10^{-15}	8.53×10^{-5}
0.4	1.36×10^{-15}	1.28×10^{-4}
0.5	1.11×10^{-15}	1.59×10^{-4}
0.6	4.27×10^{-15}	1.67×10^{-4}
0.7	5.51×10^{-15}	1.45×10^{-4}
0.8	7.28×10^{-15}	9.47×10^{-5}
0.9	1.29×10^{-15}	4.09×10^{-5}

Table 10. showing the comparison of absolute error for BVM and Cubic B-Splne in [10] for $h = 0.1$

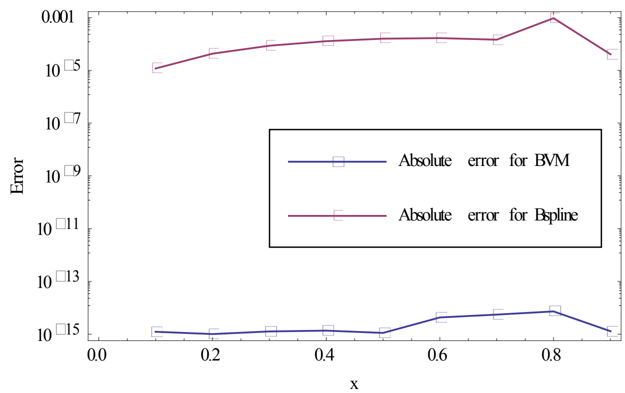

Fig. 4. Graph of error obtained for BVM and Cubic B-spline [10] for $h = 0.1$

Table 11. Comparison of maximum absolute errors for example 3

h	$1/8$	$1/16$	$1/32$
BVM	2.974×10^{-15}	2.570×10^{-17}	2.3855×10^{-17}
Pooja and Talat Sixth-order [21]	1.74×10^{-11}	8.11×10^{-13}	9.77×10^{-12}
Siddiqi and Akram [20]	1.37×10^{-6}	1.08×10^{-7}	2.25×10^{-8}

Considering different step sizes, the maximum errors obtained for Examples 1-3 shows clearly that the BVM compare favourably with the methods in the cited papers.

Fig. 3 shows that the numerical solutions compares favourably with the exact solution for h=0.1. Fig. 4 shows the comparison of absolute error obtained for Example 3 with the BVM and the method in [10]. This evidently shows that the BVM performs better thatn those compared with.

4 Conclusion

The Boundary Value Method proposed in this work was applied to solve sixth-order linear and nonlinear boundary value problems. This method has been shown to be efficient in terms of its applicability and as well as maximum the global errors obtained in the examples presented. The comparison of this method with other existing ones shows that it has compares favourably and its thus recommended for solution of general sixth order BVPs.

Competing Interests

Authors have declared that no competing interests exist.

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