



The Inverse Sushila Distribution: Properties and Application

A. A. Adetunji^{*}, J. A. Ademuyiwa¹ and O. A. Adejumo²

¹Department of Statistics, Federal Polytechnic, Ile-Oluji, Nigeria.

²Department of Statistics, University of Ilorin, Nigeria.

Authors' contributions

All authors contributed immensely to the development of the article at all stages of the formation.
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Abstract

In this paper, a new lifetime distribution called the Inverse Sushila Distribution (ISD) is proposed. Its fundamental properties like the density function, distribution function, hazard rate function, survival function, cumulative hazard rate function, order statistics, moments, moments generating function, maximum likelihood estimation, quantiles function, Rényi entropy and stochastic ordering are obtained. The distribution offers more flexibility in modelling upside-down bathtub lifetime data. The proposed model is applied to a lifetime data and its performance is compared with some other related distributions.

Keywords: Inverse Sushila distribution; Inverse Lindley distribution; lifetime data.

1 Introduction

Among numerous branches of statistics is the survival and reliability analysis. The branch has usefulness in engineering, health, demography and in industries. Time to failure of different systems can be seen as random variables and hence the relevance of modelling data from such system.

^{*}Corresponding author: E-mail: adeadetunji@fedpolel.edu.ng;

Several lifetime models pervade literatures and improvements are always made to improve efficiency, flexibility and the fits for lifetime data. The probability distribution (1) of the one-parameter Lindley distribution, Lindley [1] which is a close form of the more popular exponential distribution (2) has received little attention until recently. This is due to the recent trends in general applicability of lifetime data in reliability engineering and survival analysis. Ghitany, Atieh, & Nadarajah [2] and Krishna & Kumar [3] characterised (1) and concluded that it outperforms (2) due to its flexibility and general applicability.

$$f(x|\theta) = \frac{\theta^2}{1+\theta} (1+x)e^{-\theta x}; \quad x, \theta > 0 \tag{1}$$

$$f(x|\theta) = \theta e^{-\theta x}; \quad x, \theta > 0 \tag{2}$$

Shanker et al. [4] proposed a two-parameter (λ and θ) distribution named the ‘‘Sushila distribution’’ with (1) as special case. Moments, failure rate function, mean residual life function and stochastic ordering of the distribution were also discussed. The Sushila distribution is a mixed distribution between exponential distribution $\left(\frac{\theta}{\lambda}\right)$ and gamma distribution $\left(2, \frac{\theta}{\lambda}\right)$ with the probability distribution function and cumulative distribution function given as (3).

$$f(x|\lambda, \theta) = \frac{\theta^2}{\lambda(1+\theta)} \left(1 + \frac{x}{\lambda}\right) e^{-\frac{\theta}{\lambda}x}; \quad x, \lambda, \theta > 0 \tag{3}$$

$$F(x|\lambda, \theta) = 1 - \frac{\lambda(1+\theta)\theta x}{\lambda(1+\theta)} e^{-\frac{\theta}{\lambda}x}; \quad x, \lambda, \theta > 0 \tag{4}$$

The Inverse Sushila Distribution (ISD):

The distribution is a 2-parameter mixture of inverse Gamma distribution with (scale= $\frac{\theta}{\lambda}$) and inverse Exponential distribution with (shape=2 and scale= $\frac{\theta}{\lambda}$), with mixing proportion $p = \frac{\theta}{(\theta+\lambda)}$.

A random variable X follows Inverse Sushila Distribution is denoted with $X \sim ISD(\lambda, \theta)$ with the *pdf* and *cdf* given as (5) and (6)

$$f(x) = \frac{\theta^2}{\lambda^2(1+\theta)} \left(\frac{1+\lambda x}{x^3}\right) e^{-\frac{\theta}{\lambda}x}; \quad x, \lambda, \theta > 0 \tag{5}$$

$$F(x) = \left(1 + \frac{\theta}{\lambda x(1+\theta)}\right) e^{-\frac{\theta}{\lambda}x}; \quad x, \lambda, \theta > 0 \tag{6}$$

Note: If $\lambda = 1$, the ISD becomes the Inverse Lindley distribution by Sharma et al. [5].

The first derivative of (5) is obtained as:

$$f'(x) = -\frac{\theta^2}{\lambda^3(1+\theta)} \left(\frac{2\lambda^2 x^2 - \lambda\theta x + 3\lambda x - \theta}{x^5}\right) e^{-\frac{\theta}{\lambda}x} \tag{7}$$

Equation (7) is a unimodal function that attains its maximum value at 0. Hence, the mode of $f(x)$ is given in (8). This is obtained by equating (7) to 0 and isolating solution for x .

$$M_0 = \frac{\theta - 3 - \sqrt{(\theta - 3)^2 + 8\theta}}{4\lambda} \tag{8}$$

This article is sectionalized as follows: Section 1 introduces the new distribution along with its *pdf* and the *cdf*. Section 2 investigates the reliability functions of the ISD. These include the hazard rate, survival rate, and the cumulative hazard rate. The third section investigates various mathematical properties of the ISD

along with their respective proofs. These include the Order statistics, moments and moment generating functions, maximum likelihood estimation, Rényi entropy, stochastic ordering, and quantiles function. The fourth sections presents some related distribution to the proposed distribution and compared the performance of the new distribution with some of the related ones that pervade literatures. The final section concludes based on the findings from the research.

2 Reliability Function for ISD

If a random variable $X \sim ISD(\lambda, \theta)$, the survival function, the hazard (failure) rate function, and cumulative hazard rate function are given in (9), (10) and (11) respectively.

$$S(x) = 1 - \frac{(\lambda\theta x + \lambda x + \theta)}{\lambda x(1 + \theta)} e^{-\frac{\theta}{\lambda x}} \tag{9}$$

$$h(x) = \frac{\theta^2(1 + \lambda x)}{\lambda^2(1 + \theta)x^3 \left[1 - \frac{(\lambda\theta x + \lambda x + \theta)}{\lambda x(1 + \theta)} e^{-\frac{\theta}{\lambda x}} \right]} e^{-\frac{\theta}{\lambda x}} \tag{10}$$

$$H(x) = -\ln \left(1 - \frac{(\lambda\theta x + \lambda x + \theta)}{\lambda x(1 + \theta)} e^{-\frac{\theta}{\lambda x}} \right) \tag{11}$$

Plots of the pdf, cdf, hrf, and survival function of the ISD distribution for some selected parameter values are shown in the figures below.

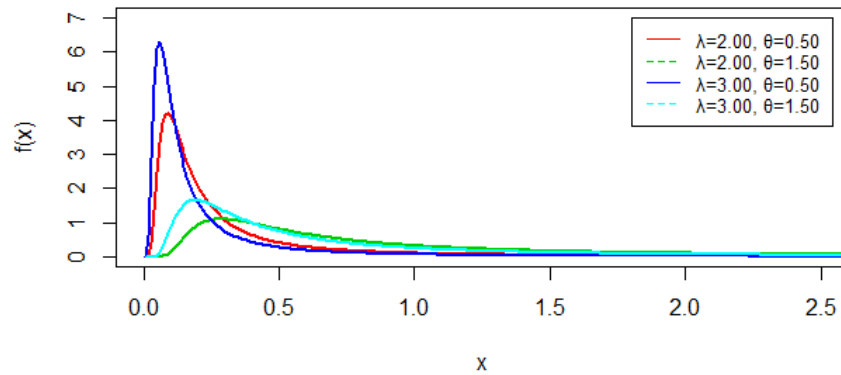


Fig. 1. Plots of the pdf of the ISD for different values of λ and θ

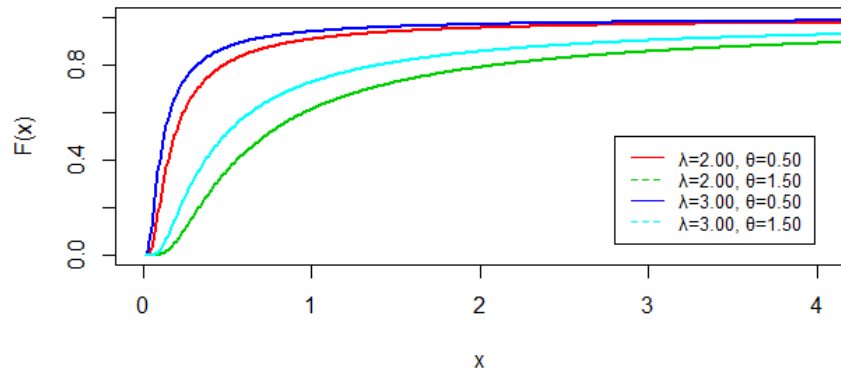


Fig. 2. Plots of the cdf of the ISD for different values of λ and θ

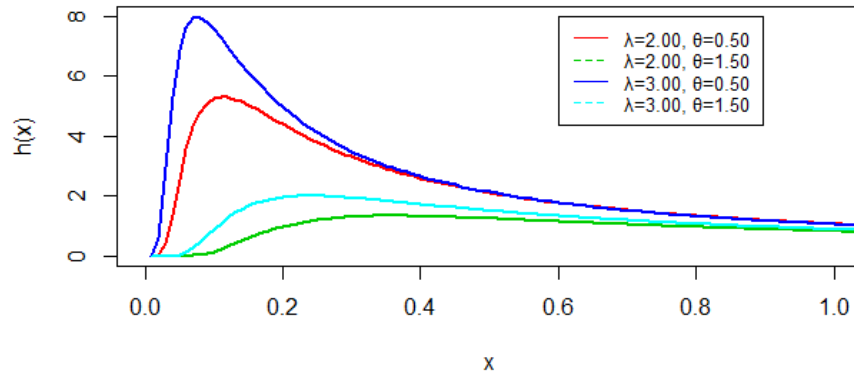


Fig. 3. Plots of the hazard rate function (*hrf*) of the ISD for different values of λ and θ

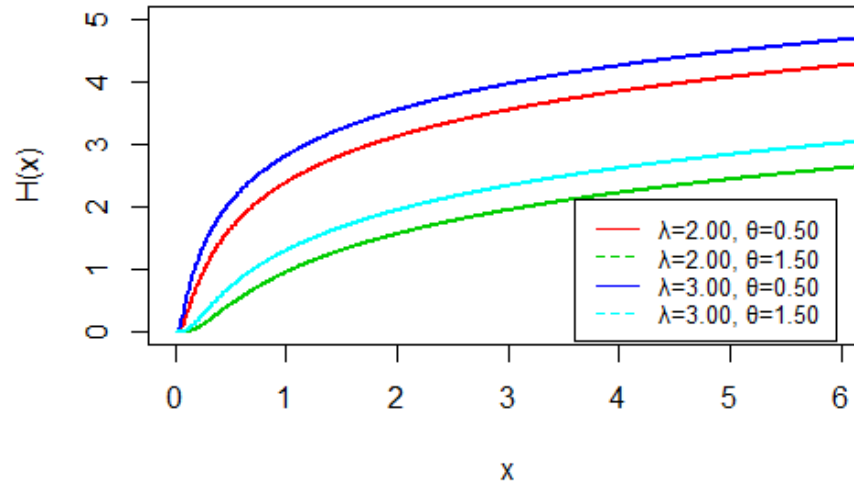


Fig. 4. Plots of the *Cumulative hrf* of the ISD for different values of λ and θ

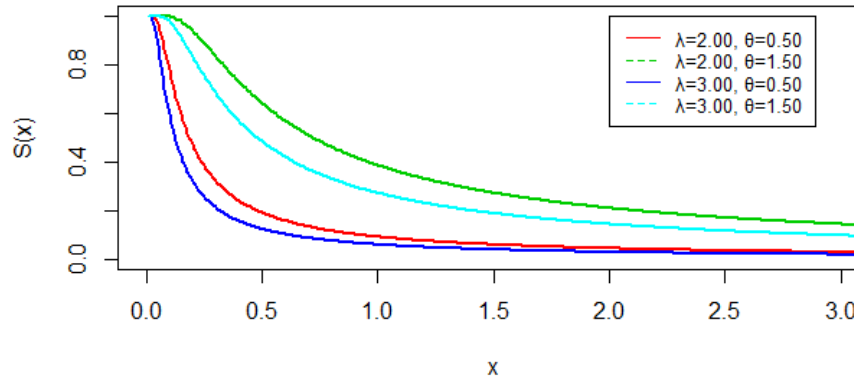


Fig. 5. Plots of the *survival function* of the ISD for different values of λ and θ

3 Properties of the Inverse Sushila Distribution

3.1 Order statistics

Let $X_{1,n} < X_{2,n} < \dots < X_{n,n}$ be the order statistics of a random sample of size n with any distribution, then if $X \sim ISD(\lambda, \theta)$, then the *pdf* of the k^{th} order statistics is given as:

$$f_{k,n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)[F(x)]^{k-1}[1 - F(x)]^{n-k}$$

Hence, the k^{th} order statistics of random variable $X \sim ISD(\lambda, \theta)$ is given as (12)

$$f_{k,n}(x) = \frac{n! \theta^2 (1+\lambda x) e^{-\frac{\theta}{\lambda x}} \left[\frac{(\lambda \theta x + \lambda x + \theta)}{\lambda x (1+\theta)} e^{-\frac{\theta}{\lambda x}} \right]^{k-1} \left[1 - \frac{(\lambda \theta x + \lambda x + \theta)}{\lambda x (1+\theta)} e^{-\frac{\theta}{\lambda x}} \right]^{n-k}}{(k-1)!(n-k)! \lambda^2 (1+\theta) x^3} \tag{12}$$

For $k = 1$ and $k = n$, the 1^{st} and n^{th} order statistic for $X \sim ISD(\lambda, \theta)$ are respectively given as (13) and (14).

$$f_{1,n}(x) = \frac{n! \theta^2 (1+\lambda x) e^{-\frac{\theta}{\lambda x}} \left[1 - \frac{(\lambda \theta x + \lambda x + \theta)}{\lambda x (1+\theta)} e^{-\frac{\theta}{\lambda x}} \right]^{n-1}}{(n-1)! \lambda^2 (1+\theta) x^3} \tag{13}$$

$$f_{n,n}(x) = \frac{n! \theta^2 (1+\lambda x) e^{-\frac{\theta}{\lambda x}} \left[\frac{(\lambda \theta x + \lambda x + \theta)}{\lambda x (1+\theta)} e^{-\frac{\theta}{\lambda x}} \right]^{n-1}}{(n-1)! \lambda^2 (1+\theta) x^3} \tag{14}$$

3.2 Moments and moments generating function

Proposition 1: Let $X \sim ISD(\lambda, \theta)$ with *pdf* as given in (4), then the r^{th} raw moment (moment about the origin) is given by (15) while the moment generating function, *MGF*, is given by (16).

$$\mu'_r = \left(\frac{\theta}{\lambda}\right)^r \frac{(1-r+\theta)}{(1+\theta)} \Gamma(1-r) \tag{15}$$

$$M_x(t) = E(e^{tx}) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{\theta}{\lambda}\right)^n \frac{(1-n+\theta)}{(1+\theta)} \Gamma(1-n) \tag{16}$$

Proof of Moments about mean

$$\begin{aligned} \mu'_r &= E(x^r) = \int_{-\infty}^{\infty} x^r f(x) dx \\ &= \int_0^{\infty} x^r \frac{\theta^2}{\lambda^2 (1+\theta)} \left(\frac{1+\lambda x}{x^3}\right) e^{-\frac{\theta}{\lambda x}} dx \\ &= \frac{\theta^2}{\lambda^2 (1+\theta)} \left[\int_0^{\infty} x^{r-3} e^{-\frac{\theta}{\lambda x}} dx + \int_0^{\infty} \lambda x^{r-2} e^{-\frac{\theta}{\lambda x}} dx \right] \end{aligned}$$

Let $y = \lambda x$ and $x = \frac{y}{\lambda}$, then $dx = \frac{1}{\lambda} dy$. Therefore,

$$\begin{aligned} \mu'_r &= \frac{\theta^2}{\lambda^2(1+\theta)} \left[\int_0^\infty \frac{y^{r-3}}{\lambda^{r-3}} e^{-\frac{\theta}{y}} \frac{1}{\lambda} dy + \lambda \int_0^\infty \frac{y^{r-2}}{\lambda^{r-2}} e^{-\frac{\theta}{y}} \frac{1}{\lambda} dy \right] \\ &= \frac{\theta^2}{\lambda^2(1+\theta)} \left[\frac{1}{\lambda^{r-2}} \int_0^\infty \frac{1}{y^{3-r}} e^{-\frac{\theta}{y}} dy + \frac{1}{\lambda^{r-2}} \int_0^\infty \frac{1}{y^{2-r}} e^{-\frac{\theta}{y}} dy \right] \end{aligned}$$

Recall that if a random variable X has Inverse Gamma distribution, then:

$$\int_0^\infty y^{-(a+1)} e^{-\frac{\theta}{y}} dy = \frac{\Gamma(a)}{\theta^a}$$

Therefore,

$$\begin{aligned} \mu'_r &= \frac{\theta^2}{\lambda^2(1+\theta)} \frac{1}{\lambda^{r-2}} \left[\frac{\Gamma(2-r)}{\theta^{2-r}} + \frac{\Gamma(1-r)}{\theta^{1-r}} \right] \\ &= \frac{\theta^2}{\lambda^2(1+\theta)} \frac{1}{\lambda^{r-2}} \left[\frac{(1-r)\Gamma(1-r) + \theta\Gamma(1-r)}{\theta^{2-r}} \right] \\ &= \left(\frac{\theta}{\lambda}\right)^r \frac{(1-r+\theta)}{(1+\theta)} \Gamma(1-r) \end{aligned}$$

Proof of Moments Generating Function

$$\begin{aligned} M_x^t &= E(e^{tx}) = \int_{-\infty}^\infty e^{tx} f(x) dx \\ &= \int_0^\infty e^{tx} \frac{\theta^2}{\lambda^2(1+\theta)} \left(\frac{1+\lambda x}{x^3}\right) e^{-\frac{\theta}{\lambda x}} dx \end{aligned}$$

Recall

$$e^{tx} = \sum_{n=0}^\infty \frac{t^n x^n}{n!}$$

Therefore,

$$M_x^t = \frac{\theta^2}{\lambda^2(1+\theta)} \sum_{n=0}^\infty \frac{t^n}{n!} \int_0^\infty \left(x^{n-3} e^{-\frac{\theta}{\lambda x}} + \lambda x^{n-2} e^{-\frac{\theta}{\lambda x}} \right) dx$$

Let $y = \lambda x$ and $x = \frac{y}{\lambda}$, then $dx = \frac{1}{\lambda} dy$

Hence,

$$\begin{aligned} M_x^t &= \frac{\theta^2 \lambda^{-n}}{(1+\theta)} \sum_{n=0}^\infty \frac{t^n}{n!} \left(\frac{\Gamma(2-n) + \theta\Gamma(1-n)}{\theta^{2-n}} \right) \\ &= \frac{\theta^n}{(1+\theta)} \lambda^{-n} \sum_{n=0}^\infty \frac{t^n}{n!} (1+\theta-n) \Gamma(1-n) \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\frac{\theta}{\lambda}\right)^n \frac{(1-n+\theta)}{(1+\theta)} \Gamma(1-n)$$

3.3 Rényi Entropy (RE)

Rényi entropy measures the variation of uncertainty in the random variable by providing tools to indicate varieties in the distribution and analyze evolutionary processes in the distribution over time, Alkarni [6]. Rényi [7] gave the expression for the entropy as:

$$\text{Re}(\pi) = \left(\frac{1}{1-\pi}\right) \log \left\{ \int f^\pi(x) dx \right\}; \quad \pi > 0$$

Proposition 2: If $X \sim ISD(\lambda, \theta)$, the RE is given as:

$$\text{Re}(\pi) = \left(\frac{1}{1-\pi}\right) \log \left\{ \sum_{i=0}^n \binom{\pi}{i} \frac{\theta^{1-i}}{(1+\theta)^\pi} \frac{\lambda^{\pi-1}}{\pi^{2\pi+i-1}} \Gamma(2\pi+i-1) \right\}; \quad \pi \neq 1 \tag{17}$$

Proof of Rényi Entropy

$$\text{Re}(\pi) = \left(\frac{1}{1-\pi}\right) \log \left\{ \int f^\pi(x) dx \right\}$$

$$\begin{aligned} \text{Re}(\pi) &= \left(\frac{1}{1-\pi}\right) \log \left\{ \int \left(\frac{\theta^2}{\lambda^2(1+\theta)} \left(\frac{1+\lambda x}{x^3}\right) e^{-\frac{\theta}{\lambda x}} \right)^\pi dx \right\} \\ &= \left(\frac{1}{1-\pi}\right) \log \left\{ \frac{\theta^{2\pi}}{\lambda^{2\pi}(1+\theta)^\pi} \int_0^\infty (1+\lambda x)^\pi x^{-3\pi} e^{-\frac{\pi\theta}{\lambda x}} dx \right\} \end{aligned}$$

Recall

$$\begin{aligned} (1+\lambda x)^\pi &= \sum_{i=0}^{\infty} \binom{\pi}{i} (\lambda x)^{\pi-i} = \sum_{i=0}^{\infty} \lambda^{\pi-i} x^{\pi-i} \\ &= \left(\frac{1}{1-\pi}\right) \log \left\{ \frac{\theta^{2\pi}}{\lambda^{2\pi}(1+\theta)^\pi} \int_0^\infty \sum_{i=0}^{\infty} \binom{\pi}{i} \lambda^{\pi-i} x^{\pi-i} x^{-3\pi} e^{-\frac{\pi\theta}{\lambda x}} dx \right\} \\ &= \left(\frac{1}{1-\pi}\right) \log \left\{ \frac{\theta^{2\pi}}{\lambda^{2\pi}(1+\theta)^\pi} \sum_{i=0}^{\infty} \binom{\pi}{i} \lambda^{\pi-i} \int_0^\infty x^{-2\pi-i} e^{-\frac{\pi\theta}{\lambda x}} dx \right\} \end{aligned}$$

Let $y = \lambda x$ and $x = \frac{1}{\lambda} y$, then $dx = \frac{1}{\lambda} dy$

$$\begin{aligned} &= \left(\frac{1}{1-\pi}\right) \log \left\{ \frac{\theta^{2\pi}}{\lambda^{2\pi}(1+\theta)^\pi} \sum_{i=0}^{\infty} \binom{\pi}{i} \lambda^{\pi-i} \int_0^\infty \left(\frac{1}{\lambda}\right)^{-2\pi-i+1} y^{-2\pi-i} e^{-\frac{\pi\theta}{y}} dy \right\} \\ &= \left(\frac{1}{1-\pi}\right) \log \left\{ \frac{\theta^{2\pi}}{\lambda^{2\pi}(1+\theta)^\pi} \sum_{i=0}^{\infty} \binom{\pi}{i} \lambda^{\pi-i} \left(\frac{1}{\lambda}\right)^{-2\pi-i+1} \int_0^\infty y^{-(2\pi+i-1+1)} e^{-\frac{\pi\theta}{y}} dy \right\} \\ &= \left(\frac{1}{1-\pi}\right) \log \left\{ \frac{\theta^{2\pi}}{\lambda^{2\pi}(1+\theta)^\pi} \sum_{i=0}^{\infty} \binom{\pi}{i} \lambda^{\pi-i} \left(\frac{1}{\lambda}\right)^{-2\pi-i+1} \frac{\Gamma(2\pi+i-1)}{(\pi\theta)^{2\pi+i-1}} \right\} \end{aligned}$$

$$\operatorname{Re}(\pi) = \left(\frac{1}{1-\pi} \right) \log \left\{ \sum_{i=0}^n \binom{\pi}{i} \frac{\theta^{1-i}}{(1+\theta)^\pi} \frac{\lambda^{\pi-1}}{\pi^{2\pi+i-1}} \Gamma(2\pi+i-1) \right\}; \pi \neq 1$$

3.4 Stochastic ordering

Stochastic ordering is used to judge comparative behaviour of continuous random variable. A random variable X is said to be stochastically smaller than another random variable Y ($Y \leq_{st} X$) in the:

- (i) Stochastic order ($X \leq_{st} Y$) if $F_X(x) \leq F_Y(x)$ for all x .
- (ii) Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \leq h_Y(x)$ for all x .
- (iii) Mean residual life order ($X \leq_{mrl} Y$) if $m_X(x) \leq m_Y(x)$ for all x .
- (iv) Likelihood rate order ($X \leq_{lr} Y$) if $\frac{f_X(x)}{f_Y(x)}$ for all x is an decreasing function in x .

The following theorem shows that $X \sim ISD(\lambda, \theta)$ is ordered with respect to “likelihood ratio” ordering according to the model by Shaked & Shanthikumar [8].

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \Rightarrow X \leq_{st} Y$$

Proposition 3: Let $X \sim ISD(\lambda_1, \theta_1)$ and $Y \sim ISD(\lambda_2, \theta_2)$. If $\theta_2 \geq \theta_1$ and $\lambda_1 = \lambda_2$ (or if $\theta_1 = \theta_2$ and $\lambda_2 \geq \lambda_1$), then $X \leq_{lr} Y$. By the implication of the Shaked & Shanthikumar model, hence, $X \leq_{mrl} Y$, $X \leq_{hr} Y$, and $X \leq_{st} Y$.

Proof

$$\begin{aligned} \frac{f_X(x)}{f_Y(x)} &= \frac{\frac{\theta_1^2}{\lambda_1^2(1+\theta_1)} \left(\frac{1+\lambda_1 x}{x^3} \right) e^{-\frac{\theta_1}{\lambda_1 x}}}{\frac{\theta_2^2}{\lambda_2^2(1+\theta_2)} \left(\frac{1+\lambda_2 x}{x^3} \right) e^{-\frac{\theta_2}{\lambda_2 x}}} \\ &= \frac{\theta_1^2(1+\lambda_1 x)\lambda_2^2(1+\theta_2)}{\theta_2^2(1+\lambda_2 x)\lambda_1^2(1+\theta_1)} e^{-\frac{\lambda_2\theta_1+\lambda_1\theta_2}{\lambda_1\lambda_2 x}} \end{aligned}$$

For $\lambda_1 = \lambda_2$ and $\theta_2 \geq \theta_1$ (or if $\theta_1 = \theta_2$ and $\lambda_2 \geq \lambda_1$), then $\frac{f_X(x)}{f_Y(x)}$ is a decreasing function in x . By implication, $X \leq_{lr} Y$. Hence, $X \leq_{mrl} Y$, $X \leq_{hr} Y$, and $X \leq_{st} Y$.

3.5 Quantiles function

Given that $X \sim ISD(\lambda, \theta)$, obtaining formulas for the mean and the variance are quite tasking, because of complex algebraic expressions which involved integrals, the quantiles are quite easy to evaluate.

Proposition 4: Let $X \sim ISD(\lambda, \theta)$, then the quantiles function of X , say $Q_{(p)}$ is given as:

$$Q_{(p)} = - \frac{\theta}{\lambda(W[-e^{-(1+\theta)}(1+\theta)p] + 1)} \tag{18}$$

Hence, the first, second (the median), and the third quantiles of $X \sim ISD(\lambda, \theta)$ are respectively given as:

$$Q_{(0.25)} = - \frac{\theta}{\lambda(W[-0.25 e^{-(1+\theta)}(1+\theta)] + \lambda(1+\theta))}$$

$$Q_{(0.50)} = -\frac{\theta}{\lambda(W[-0.50 e^{-(1+\theta)}(1+\theta)] + \lambda(1+\theta))}$$

$$Q_{(0.7)} = -\frac{\theta}{\lambda(W[-0.75 e^{-(1+\theta)}(1+\theta)] + \lambda(1+\theta))}$$

Proof

Since $Q_{(p)} = F^{-1}(p), p \in (0,1)$. This implies that $F(Q_{(p)}) = p$. Substituting $Q_{(p)}$ for x in (5) and solve for $Q_{(p)}$.

$$F(Q_{(p)}) = \frac{(\lambda\theta Q_{(p)} + \lambda Q_{(p)} + \theta)}{\lambda Q_{(p)}(1+\theta)} e^{-\frac{\theta}{\lambda Q_{(p)}}} = p$$

$$\frac{(\lambda\theta Q_{(p)} + \lambda Q_{(p)} + \theta)}{\lambda Q_{(p)}(1+\theta)} e^{-\frac{\theta}{\lambda Q_{(p)}}} = p$$

Multiplying both sides of the equation by $-e^{-(1+\theta)}$ gives:

$$\frac{(\lambda\theta Q_{(p)} + \lambda Q_{(p)} + \theta)}{\lambda Q_{(p)}(1+\theta)} e^{-\left(\frac{\theta}{\lambda Q_{(p)}} + 1 + \theta\right)} = -pe^{-(1+\theta)}$$

$$\lambda\theta Q_{(p)} e^{-\left(\frac{\theta}{\lambda Q_{(p)}} + 1 + \theta\right)} + \lambda Q_{(p)} e^{-\left(\frac{\theta}{\lambda Q_{(p)}} + 1 + \theta\right)} + \theta e^{-\left(\frac{\theta}{\lambda Q_{(p)}} + 1 + \theta\right)} = -p(\lambda Q_{(p)} + \theta) e^{-(1+\theta)}$$

where $-(\lambda Q_{(p)} + \theta) e^{-(1+\theta)}$ is the **Lambert W** function of the real argument $-pe^{-(1+\theta)}$.

The Lambert function is a complex function with multiple values which is defined as the solution for the equation $W(u)e^{W(u)} = u$, where u is a complex number. Hence, by isolating solution for $Q_{(p)}$, obtained (18).

$$Q_{(p)} = -\frac{\theta}{\lambda(W[-pe^{-(1+\theta)}(1+\theta)] + \lambda(1+\theta))} = -\frac{\theta}{\lambda(W[-e^{-(1+\theta)}(1+\theta)p] + 1)}$$

3.6 Parameter estimation

Theorem 1.5: If X_1, X_2, X_3, \dots are independently and identically distributed random variables of size n from an $X \sim ISD(\lambda, \theta)$, then the likelihood function of X is defined as (19).

$$\log L = 2n \log(\theta) - 2n \log(\lambda) - n \log(1 + \theta) + \sum_{i=1}^n \log(1 + \lambda x_i) - 3 \sum_{i=1}^n \log(x_i) - \frac{\theta}{\lambda} \sum_{i=1}^n \frac{1}{x_i} \tag{19}$$

Proof of MLE

$$\begin{aligned} L(\theta, \lambda|x) &= \prod_{i=1}^n \frac{\theta^2}{\lambda^2(1+\theta)} (1 + \lambda x_i) x_i^{-3} e^{-\frac{\theta}{\lambda x_i}} \\ &= \frac{\theta^{2n}}{\lambda^{2n}(1+\theta)^n} \prod_{i=1}^n (1 + \lambda x_i) x_i^{-3} e^{-\frac{\theta}{\lambda x_i}} \end{aligned}$$

$$\begin{aligned} \log L(\theta, \lambda|x) &= \log \left(\prod_{i=1}^n \frac{\theta^2}{\lambda^2(1+\theta)} (1+\lambda x_i) x_i^{-3} e^{-\frac{\theta}{\lambda x_i}} \right) \\ &= 2n \log(\theta) - 2n \log(\lambda) - n \log(1+\theta) + \sum_{i=1}^n \log(1+\lambda x_i) - 3 \sum_{i=1}^n \log(x_i) - \frac{\theta}{\lambda} \sum_{i=1}^n \frac{1}{x_i} \end{aligned}$$

The MLEs ($\hat{\lambda}$ and $\hat{\theta}$) of λ and θ are respectively obtained by solving the nonlinear equations $\sum_i(x|\theta, \lambda) = 0$ where $\sum_i(x|\theta)$ and $\sum_i(x|\lambda)$ respectively represent the partial derivatives of the log-likelihood function derived in (19). For hypothesis testing and interval estimation of the two parameters, the observed information matrix $I_n(\Phi)$ is required where:

$$I_n(\Phi) = \begin{bmatrix} \frac{\partial \log L}{\partial \lambda} & \frac{\partial^2 \log L}{\partial \theta \partial \lambda} \\ \frac{\partial^2 \log L}{\partial \theta \partial \lambda} & \frac{\partial^2 \log L}{\partial \theta} \end{bmatrix}$$

Where

$$\begin{aligned} \frac{\partial \log L}{\partial \lambda} &= -\frac{n}{\lambda} + \frac{S_1}{n + \lambda S_1} + \frac{\theta S_2}{\lambda^2} \\ \frac{\partial \log L}{\partial \theta} &= \frac{2n}{\theta} - \frac{n}{1+\theta} - \frac{S_2}{\lambda} \\ \frac{\partial^2 \log L}{\partial \theta \partial \lambda} &= \frac{S_2}{\lambda^2} \\ S_1 &= \sum_{i=1}^n x_i \\ S_2 &= \sum_{i=1}^n \frac{1}{x_i} \end{aligned}$$

4 Applications

The data on flood levels for the river Susquehanna in Pennsylvania over a 20 year period had been used to compare performances of distributions related to Inverse Sushila. The data was first reported by Dumonceaux & Antle [9]. Using R language (R Core Team [10]), the distributions in Table 1 below are compared with the proposed distribution and the results obtained are presented in Table 2. Oguntunde et al. [11] proposed the Kumaraswamy Inverse Exponential (KIE) distribution while the Inverse Exponential (IE) distribution was proposed by Keller & Kamth [12]. Sharma et al. [5] proposed the Inverse Lindley (IL) distribution while the Extended Inverse Lindley (EIL) distribution was introduced by Alkarn [6].

Table 1. Distribution functions of compare models

Distribution	pdf	cdf
Kum.. Inverse Exponential (KIE)	$ab \left(\frac{\theta}{x^2}\right) \left(e^{-\frac{\theta}{x}}\right)^a \left(1 - \left(e^{-\frac{\theta}{x}}\right)^a\right)^{b-1}$	$1 - \left(1 - \left(e^{-\frac{\theta}{x}}\right)^a\right)^b$
Inverse Exponential (IE)	$\frac{\theta}{x^2} \left(e^{-\frac{\theta}{x}}\right)$	$e^{-\frac{\theta}{x}}$
Inverse Lindley (IL)	$\frac{\theta^2}{1+\theta} \left(\frac{1+x}{x^3}\right) e^{-\frac{\theta}{x}}$	$\left(1 + \frac{\theta}{(1+\theta)x}\right) e^{-\frac{\theta}{x}}$
Extended Inverse Lindley (EIL)	$\frac{\lambda \theta^2}{\beta+\theta} \left(\frac{\beta+x\lambda}{x^{2\lambda+1}}\right) e^{-\frac{\theta}{x\lambda}}$	$\left(1 + \frac{\theta \beta}{(\beta+\theta)x\lambda}\right) e^{-\frac{\theta}{x\lambda}}$

Table 2. Parameter estimates, Log-likelihood and AIC of competing models

	Par1	Par2	Par3	Log-likelihood	AIC	Rank
ISD	0.0000542	0.0000428		4.291	-4.583	2
KIE	0.266	37.699	6.177	15.411	-24.822	1
IE	0.394			-2.740	7.481	4
IL	0.635			0.585	0.829	3
EIL	0.580	1.964	479.635	-120.421	246.842	5

Table 2 shows the summary of the result using the flood data for the proposed ISD and other related distributions. It can be observed that only KIE outperforms the ISD. This may not be unconnected with the fact that the KIE is a compounded distribution with an extra parameter and hence extra flexibility.

5 Conclusion

A new two-parameter distribution called the Inverse Sushila Distribution (ISD). The distribution is a mixture of Inverse Gamma and Inverse Exponential distributions. The new distribution offers flexibility in modelling upside-down bathtub lifetime data. Various properties of the new distribution (including the pdf, cdf, hazard rate function, survival function, cumulative hazard rate function, order statistics, moments, moment generating function, maximum likelihood estimation of parameters, quantile function, Rényi entropy and stochastic ordering are obtained. The performance of the new distribution is compared with some related distribution using real life data and the result showed that it competes favourably.

Competing Interests

Authors have declared that no competing interests exist.

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