British Journal of Mathematics & Computer Science
12(4): 1-15, 2016, Article no.BJMCS.20083
ISSN: 2231-0851

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The Motion Equation of a Spring-Magnet-Mass System Placed in Nonlinear Magnetic Field. An Analytical Solution of Elliptic Sine form Functions

Nicusor Nistor[1*], Constantin Gheorghies[1] and Nelu Cazacu[2]

[1]Department of Physics, Chemistry and Environment, Faculty of Science and Environment, Dunarea de Jos University of Galati, Romania.
[2]Department of Materials Science and Engineering, Faculty of Engineering, Dunarea de Jos University of Galati, Romania.

Article Information

DOI: 10.9734/BJMCS/2016/20083
Editor(s):
(1) Paul Bracken, Department of Mathematics, The University of Texas-Pan American, Edinburg, TX 78539, USA.
Reviewers:
(1) Xinjie Li, Wuhan University, Wuhan, China.
(2) Haiduke Sarafian, The Pennsylvania State University, USA.
Complete Peer review History: http://sciencedomain.org/review-history/11861

Original Research Article

Received: 10 July 2015
Accepted: 9 September 2015
Published: 17 October 2015

Abstract

In this work, we are studying about a special oscillator system, which consists of one spring and a magnet-mass. The system is placed in nonlinear magnetic field, produced by two other permanent magnets, which are oriented for attraction, where can appear different types of oscillations. The magnet-body is simultaneously the subject of the linear field of spring and also of the nonlinear magnetic field of permanent magnets which has inverse quadratic dependence on distance. We are studying the ideal case, without friction, where the oscillations are produced with energy conservation, the oscillator system is started by applying the initial impulse and we consider the hypothesis that magnetic field produced by the permanent magnets is conservative and there is no loss of energy in the magnetic interactions. We are going to find the law of motion for the general case of study and a typically numerical application will be done.

Keywords: Nonlinear field of forces; nonlinear differential equation; elliptic sine functions; special function prototype; degree of eccentricity.

**Corresponding author: E-mail: Nicusor.Nistor@ugal.ro;*

1 Introduction

The studied mechanical system, was inspired by an experimental activity conducted by the authors, and this article aims to find, the mathematical laws of physical motion.

1.1 Describing the Mechanical System

We start from the figure below, which describe the proposed mechanical system, where is shown a sub system of magnet-body placed in nonlinear field of magnetic forces. The permanent magnet which is inside of the m body, is oriented as shown and can move freely with it, in X axis direction.

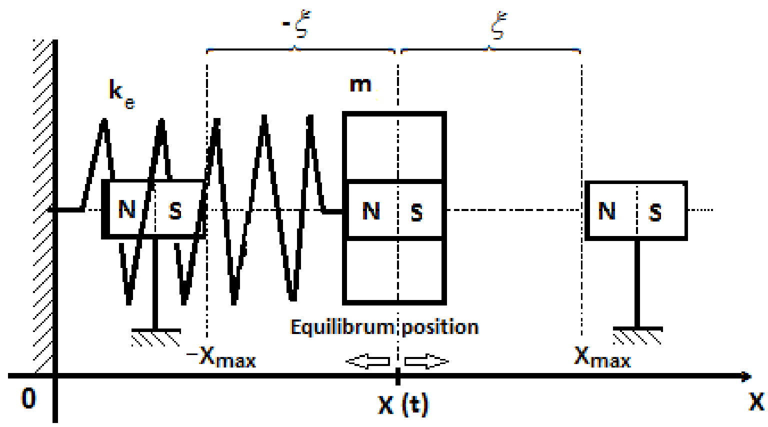

Fig. 1. Mechanical system with nonlinear field interactions

In this theoretical model the ideal expressions of forces was considered, so that for magnetic force, we used a simple monopole-monopole expression, for a long bar magnets, described in [1]. Expression of magnetic field of forces (1), is well known formula, where: F_m - is the magnetic force, λ_m - is a constant depending on type and strength of the magnets, and X - are the distance between magnets. It was considered that there is no friction or loss energy in magnetic interactions and the problem was approached using the law of conservation of energy. We observe that the force has quadratic inverse variation with distance, which gives a nonlinear variations.

$$F_m = \frac{\lambda_m}{X^2} \tag{1}$$

Considering that centrums of coordinates system is in $X = 0$, the zero point for maximum value of magnetic forces is near surfaces of magnets 1 and 2, in $-\xi$ and ξ points, like in the next figure.

The forces which acting on magnet no. 3, was expressed in (2), and has inverse variation beside deviation from the equilibrium position of the mechanical systems. The same formula was described also in the work of the other authors [2], as follows:

$$\begin{cases} F_{1m} = \dfrac{\lambda_m}{(\xi - X)^2} \\ F_{2m} = -\dfrac{\lambda_m}{(\xi + X)^2} \end{cases} \tag{2}$$

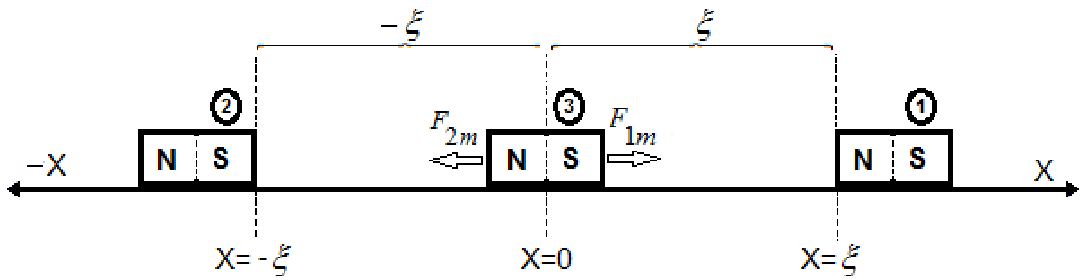

Fig. 2. Schematic displacement, of permanent magnets on X axis

The above figure shows the point of equilibrium of forces for X=0, maximum or minimum reached of forces as function of X. The magnets 1 and 2 are fixed, and will not consider the interact force between them, but only strength of the forces acting on the magnet 3, hereinafter called: F_{1m} and F_{2m}

The third magnet is subjected simultaneously to two complementary strengths, as following discussion:

- When $X=-\ \xi$, we have next situation:

$$\begin{cases} F_{1m}=\dfrac{\lambda_m}{(\xi+\xi)^2}=\dfrac{\lambda_m}{4\xi^2} \\ F_{2m}=-\dfrac{\lambda_m}{(\xi-\xi)^2} \end{cases} \tag{3}$$

Where we observe that F_{1m} has a minimum value

- When $X=0$, we have next situation:

$$F_{1m}+F_{2m}=-\frac{\lambda_m}{\xi^2}+\frac{\lambda_m}{\xi^2}=0 \tag{4}$$

- When $X=\xi$, we have next situation:

$$\begin{cases} F_{1m}=\dfrac{\lambda_m}{(\xi-\xi)^2} \\ F_{2m}=-\dfrac{\lambda_m}{(\xi+\xi)^2}=-\dfrac{\lambda_m}{4\xi^2} \end{cases} \tag{5}$$

Where we observe that F_{2m} has a minimum value.

In Fig. 2, group of: ξ, -ξ points are the maximum positive or negative displacement of the system, from the equilibrium point, and here it was considered that the distance between magnets: 1-3, 2-3, will never become zero, where the magnetic force, tends to infinity. The magnetic force model was described also in the almost same mode in [3].

1.2 The Constant of the Mechanical System are

k_e – are the elastic constant of the spring.

λ_m – are magnetic constant depending on strength of the permanent magnet.

m – are the mass of the magnet-mass body.

ξ – is a distance depending on construction of mechanical system.

1.3 The Implied Static and Dynamic Forces are Follows

F_e – Elastic force of the spring

$$F_e = -k_{el} X(t)\ \vec{i} \tag{6}$$

Fm_1 – Magnetic force of permanent magnet 1

$$F_{1m} = \frac{\lambda_m}{[\xi - X(t)]^2} \vec{i} \tag{7}$$

Fm_2 – Magnetic force of permanent magnet 2

$$F_{2m} = -\frac{\lambda_m}{[\xi + X(t)]^2} \vec{i} \tag{8}$$

F_r – The resultant of static forces:

$$F_r = -k_e X(t)\vec{i} + \frac{\lambda_m}{[\xi - X(t)]^2}\vec{i} - \frac{\lambda_m}{[\xi + X(t)]^2}\vec{i} \tag{9}$$

F_i – Dynamic forces of acceleration

$$F_i = m \frac{d^2 X(t)}{dt^2}\vec{i} \tag{10}$$

Equation of equilibrium of forces is

$$F_i = F_r \tag{11}$$

Taking into account the configuration of dynamical and statically forces in the above system, the second order nonlinear differential equation that describes the mechanical system will be following:

$$m\frac{d^2X(t)}{dt^2}=-k_eX(t)+\frac{\lambda_m}{[\xi-X(t)]^2}-\frac{\lambda_m}{[\xi+X(t)]^2} \tag{12}$$

1.4 Point with Low Potential Energy of the System

The static forces fields are functions of time, by dynamic point of view, but also function of position, as well. By integrating once (12), formally calculated according to $X(t)$, we get a low-order differential equation, in which the left side has physical dimension of kinetic energy of system, and right side expressing the potential energy.

$$\frac{m}{2}\left[\frac{dX(t)}{dt}\right]^2=\frac{-kX^2(t)}{2}+\frac{\lambda_m}{\xi-X(t)}+\frac{\lambda_m}{\xi+X(t)} \tag{13}$$

If we plot the right side of (12) and right side of (13), will obtain the figure below, where graph of the static forces -function and potential energy - function are shown.

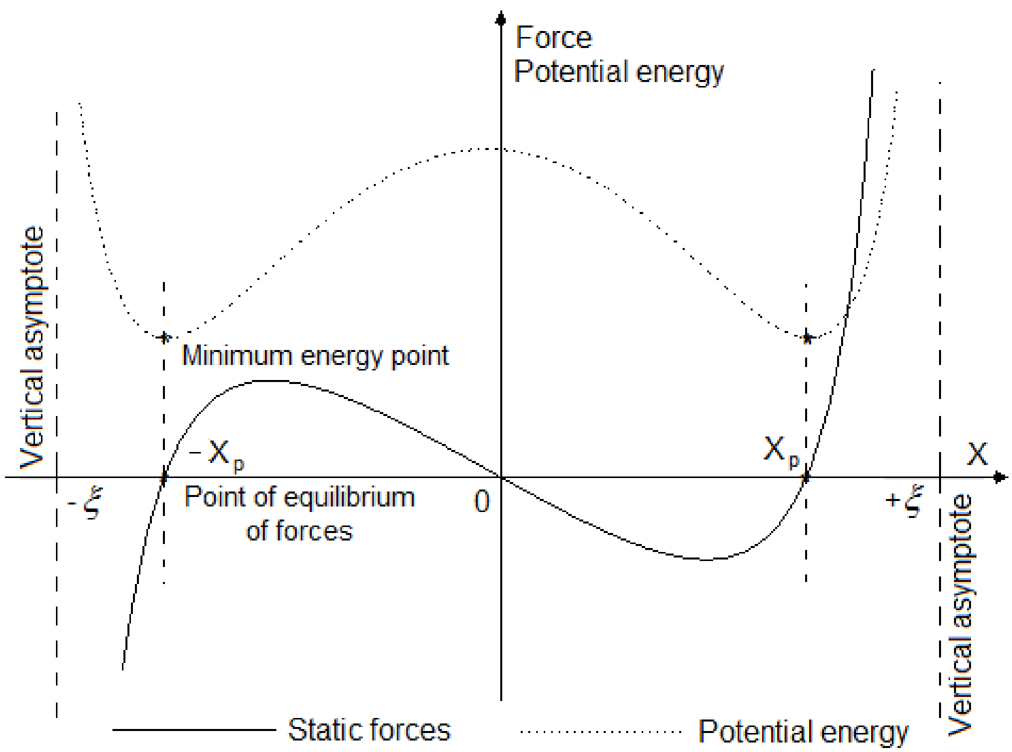

Fig. 3. Force and potential energy variation versus position. Equilibrium points

We observe that in continuity of the function domain, the force expression are passing tree time from zero, instead potential ennergy has two minimum points, where the system has stability points. In this study, we will show that our nonlinear differential equation (12) admits in these domains, two particular constant solutions, where it will be possible to observe a halt of the body, from oscillatory motion, depending on start position.

2 Applied Methods

We start to find the solution for this nonlinear differential equation, by analyzing a few special functions and we study the derivative of second order for these functions.

2.1 Jacobi Elliptic Functions like Natural Oscillation Solution of Some Nonlinear Second Order Differential Equation

Looking at a material point, being in circular motion with constant angular velocity, we can remark the following: From the point of view of X axis, we can see the variation of trigonometric cosine function, and from the point of view of Y axis, we could see the variation of trigonometric sine function. In the same way, if we are looking at the elliptical movement of a material point with constant angular velocity from the X axis point of view, the material point looks like having regular stops, by decrease of the speed of motion in time, in concordance with Jacobi SN (x, k) function and it seems to have sudden changes of speed of motion in time viewed from Y, in concordance with Jacobi CN (x, k) function.

Starting from this point, and considering a special nonlinear characteristics of the system studied in this paper, the authors will use a natural application of elliptic functions, of special metrics developed in the works of Niels Henry Abel and Carl Gustav Jacobi.

Starting for ellipse equation, of s_1 and s_2 semi axes:

$$\frac{X^2}{s_1^2}+\frac{Y^2}{s_2^2}=R^2 \tag{14}$$

Were we defining the k - term, hereinafter called, eccentricity of ellipse, with expression:

$$\mathrm{k}=\sqrt{1-\frac{s_2^2}{s_1^2}} \tag{15}$$

The length of ellipse sector, which is widely described, as is shown in [4], is:

$$dl^2=\left(dX^2+dY^2\right)d^2\varphi \tag{16}$$

By tacking:

$$\begin{cases} X=s_2\cos(\varphi) \\ Y=s_1\sin(\varphi) \end{cases} \tag{17}$$

We obtain the expression of perimeter of an ellipse:

$$dl^2=s_1^2[(\sin^2(\varphi)-\mathrm{k}^2\sin^2(\varphi))+\cos^2(\varphi)]d\varphi^2=s_1^2[(1-\mathrm{k}^2\sin^2(\varphi))]d\varphi^2 \tag{18}$$

For a normalized ellipse, if $s_1 = 1$, we find an integral formula to reach the length of an ellipse sector, as follow:

$$\int_a^b dl = \int_0^\varphi \sqrt{[(1-\mathrm{k}^2 \sin^2(\varphi))]}\, d\varphi \tag{19}$$

Starting to this integral form, based on Euler's substitutions, Carl Jacobi wrote the following remarkable result, as first incomplete elliptic integral:

$$t = \int_0^\theta \frac{d\varphi}{\sqrt{1-\mathrm{k}^2 \sin^2(\varphi)}} = \int_0^{x=\sin(\varphi)} \frac{dx}{\sqrt{1-x^2}\sqrt{1-\mathrm{k}^2 x^2}} \tag{20}$$

Where we can observe, some special function, described also in work [5,6], as follow:

The amplitude of t:

$$am(t)=\varphi \tag{21}$$

Sine amplitude of t:

$$sn(t,k)=\sin[am(t)]=\sin(\varphi) \tag{22}$$

Cosine amplitude of t:

$$cn(t,k)=\cos[am(t)]=\cos(\varphi) \tag{23}$$

Delta amplitude of t, which has no correspondent in classical trigonometric functions but it has utility in derivative of elliptic functions:

$$dn(t,k)=\sqrt{1-\mathrm{k}^2 \sin^2(\varphi)}=\sqrt{1-\mathrm{k}^2 sn^2(t,k)} \tag{24}$$

Similar form of fundamental trigonometry formula:

$$sn^2(t,k)+cn^2(t,k)=1 \tag{25}$$

The Jacobi elliptic functions are inverse solutions of elliptic integrals, and are well known their usage as functions that check some nonlinear differential equations as shown below. We start from a dedicated form of nonlinear equation of second order of this pattern:

$$\frac{dX^2(t)}{dt^2}=-(1+k^2)X(t)+2k^2X^3(t) \tag{26}$$

By substituting:

$$X(t) = sn(t,k) \tag{27}$$

And calculating the derivative of second order, of the Jacobi Sine function, we obtain:

$$\frac{d^2[sn(t,k)]}{dt^2} = \frac{d}{dt}\left\{\frac{d[sn(t,k)]}{dt}\right\} = \frac{d}{dt}[cn(t,k)dn(t,k)] = -dn^2(t,k)sn(t,k) - k^2 cn^2(t,k)sn(t,k) \tag{28}$$

Taking into account of (24) and (25), we get:

$$\frac{d^2[sn(t,k)]}{dt^2} = -\left[1 - \mathrm{k}^2 sn^2(t,k)\right]sn(t,k) - k^2\left[1 - sn^2(t,k)\right]sn(t,k) \tag{29}$$

Thus, after simplifying:

$$\frac{d^2[sn(t,k)]}{dt^2} = -sn(t,k) + \mathrm{k}^2 sn^3(t,k) - k^2 sn(t,k) + k^2 sn^3(t,k) \tag{30}$$

And finally:

$$\frac{d^2[sn(t,k)]}{dt^2} = -(1+k^2)sn(t,k) + 2\mathrm{k}^2 sn^3(t,k) \tag{31}$$

Where observe that solution is the exactly form of (27), substituted in (26).

2.2 Analytical Solution of Given Problem:

The solution of given differential equation will be a family of function depending on initial displacement of starting impulse, denoted as: x_0. A very interesting fact is this: for the classic harmonic oscillator, mass and spring, in ideal case, which has a sine solution form, the initial displacement are influencing only the maximum amplitude of motion, and phase, but locking to Jacobi Sine function, the initial displacement are influencing the time-argument also, as in the demonstration below. So first, we consider a starting position displacement from the equilibrium position, from where the motion law will have each time, different form. The time argument, it will be the subject of influencing of starting position displacement.

The new argument αt, will depend on degree of nonlinear field, and are influencing the rapidity of repetition of function. The proposed form, described also in [7], was here proposed in the form of:

$$X(t) = x_0 sn(\alpha t, k) \tag{32}$$

We start to search the solution of the nonlinear differential equation (12), by calculating the second order derivative of proposed form (32), in this way:

$$\frac{d^2\left[x_0 sn(\alpha t,k)\right]}{dt^2} = -x_0\alpha^2 dn^2(\alpha t,k)sn(\alpha t,k) - x_0\alpha^2 k^2 cn^2(\alpha t,k)sn(\alpha t,k) \quad (33)$$

After substitution (23), (24) and (25) we obtain only elliptic sine dependent form, described also in [7], as follow:

$$\frac{d^2\left[x_0 sn(\alpha t,k)\right]}{dt^2} = -x_0\alpha^2 sn(\alpha t,k) + 2x_0\alpha^2 k^2 sn^3(\alpha t,k) - x_0\alpha^2 k^2 sn(\alpha t,k) \quad (34)$$

And by applying the (32) form substitution, we get:

$$\frac{dX^2(t)}{dt^2} = -x_0\alpha^2 X(t) + 2x_0\alpha^2 k^2 X^3(t) - x_0\alpha^2 k^2 X(t) \quad (35)$$

Derivative of second order, of proposed elliptic sine form, are generating a third order algebraic polynomial, which contains both information: the argument, and eccentricity.

In this point of calculus, we have to look at the right member of (26), and observe that:

- By taking: $\mathrm{k} = 0$, will obtain:

$$\frac{dX^2(t)}{dt^2} = -X(t) \quad (36)$$

Where the solution is, the well-known form of trigonometric sine.

- By taking: $\mathrm{k}=1$, we get:

$$\frac{dX^2(t)}{dt^2} = -2X(t) + 2X^3(t) \quad (37)$$

Which it has a couple of twice-constant solution: $X(t) = \pm 1$ which seems to be a halt in time of the oscillator.

We conclude that the algebraic member of differential equations (12), will determine the amount of eccentricity of the ellipse that describes our system. Because trough the right member of (26), the differential form are reaching his particular constant solutions: $X(t) = \pm 1$, for maximum of eccentricity: $k = 1$, we choose new form of right member of (26), containing a new parameter p, to reach the same situation:

$$\mathrm{p} = \mathrm{Root}\left[-(1+\mathrm{p}^2\mathrm{k}^2)\mathrm{x_p} + 2\mathrm{p}^2\mathrm{k}^2\mathrm{x_p^3} = 0\right] \quad (38)$$

Which are solved for: x_p and k, calculated as following:

$$x_p = Root(F_r) = Root\left\{\frac{1}{m}\left[-k_{el}X + \frac{\lambda_m}{(b-X)^2} - \frac{\lambda_m}{(b+X)^2} = 0\right]\right\} \tag{39}$$

and:

$$k = k_{\max} = 1 \tag{40}$$

Where x_p is a maximum position constant of mechanical system, where equilibrium of static forces is reaching and graph of algebraic member is passing trough zero. After finding the p - parameter, we define a new eccentricity function k, depending on initial displacement, by solving for k, the equation:

$$2p^2k^2x_0^3 - (1+p^2k^2)x_0 = 0 \tag{41}$$

We find the eccentricity-argument of elliptic function, as following:

$$k(p,x_0) = \frac{1}{p\sqrt{2x_0^2 - 1}} \tag{42}$$

In order to find the new time-argument of elliptic sine function, we will find a special function prototype ad noted as - F_p, starting from our dedicated form of right member of (26), as following:

$$F_p(a,b,p,k,x_0) = a\left[-(1+p^2k^2)x_0\right] + b\left(2p^2k^2x_0\right) \tag{43}$$

Where, in finding of: a and b correction constants we have to introduce the next system of integral equations, implied in solving of our problem:

$$\begin{cases} F_p = F_r \Leftrightarrow -a(1+p^2k^2)x_p + 2bp^2k^2x_p^3 = -\dfrac{k_{el}x_p}{m} + \dfrac{\lambda_m}{m(\xi - x_p)^2} - \dfrac{\lambda_m}{m(\xi + x_p)^2} \\ \displaystyle\int_0^{x_p} F_p dX = \int_0^{x_p} F_r dX \end{cases} \tag{44}$$

With specification that first equation are solved for: x_p, p, k, being constants, and second equation are integrated for variable X, and for constants: p, k. After the above system is solved to find a, b - constants, we have to identify the coefficients of the expression, provided from right member of (35):

$$-x_0\alpha^2 X+2x_0\alpha^2 k^2 X^3-x_0\alpha^2 k^2 X=-a(1+p^2k^3)X \ \ +2bp^2k^2X^3 \tag{45}$$

Starting to the system:

$$\begin{cases} -x_0\alpha^2 - x_0\alpha^2 k^2 = -a(1+p^2k^2) \\ \qquad 2x_0\alpha^2 k^2 = 2bp^2k^2 \end{cases} \tag{46}$$

Where after solving for α, for undetermined k (the above system is a linear dependent one), we will be able to find the final form of the function.

Together (44) with (47), are compound the final solution, in the form:

$$X(t)=x_0 sn\left[\alpha(x_0,p,k,x_p)t,k(x_0,p)\right] \tag{47}$$

3 A Specific Numerical Application

Finally, we give an example to illustrate the result obtained in this paper.

Considering the following physics constants of the system, we will get the particular solution of the mechanical system:

$$k_{el}=2 \ \ mN/mm; \ \ \lambda_m=20 \ \ mN/mm; \ \ m=1 \ \ gram; \ \ \xi=5 \ \ mm; \tag{48}$$

The (9) form is becoming:

$$F_r=\left[-2X+\frac{20}{(5-X)^2}-\frac{20}{(5+X)^2}\right] \tag{49}$$

We start to find the derived system constants, as follow:

$$x_p=Root\left[-2X+\frac{20}{(5-X)^2}-\frac{20}{(5+X)^2}=0\right]=\left\{-\sqrt{25+10\sqrt{2}},\ -\sqrt{25-10\sqrt{2}},\ 0,\ \sqrt{25-10\sqrt{2}},\ \sqrt{25+10\sqrt{2}}\right\} \tag{50}$$

We choose only couple of twice solution between both vertical asymptotes, as shown in Fig. 4:

$$x_p=\pm\sqrt{25-10\sqrt{2}}=\pm \ 3.295mm \tag{51}$$

We find the p - parameter, which define the depth of nonlinearity, of our particular case, applying the (38), for: $\left|x_p\right|=3.295$, and maximum of eccentricity $k_{\max}=1$, in the follow:

$$2p^2(3.295)^3 - 3.295\ (1 + p^2) = -3.295 + 68.252p^2 = 0 \quad (52)$$

And getting:

$$p = \pm 0.219731 \Leftrightarrow |p| = 0.219731 \quad (53)$$

So that in this particular case, the eccentricity depending on initial position function is:

$$k(x_0) = \frac{1}{0.219731\ \sqrt{2x_0^2 - 1}} = \frac{4.551}{\sqrt{2x_0^2 - 1}} \quad (54)$$

Finding of a, b constants, implies the system (44) for: $|x_p| = 3.295,\ |p| = 0.219731,\ k = 1$, as follow:

$$\begin{cases} -3.45407077\ \ a + 3.45407077\ \ b = -0.00078220 \\ -5.69058160\ \ a + 2.84529080\ \ b = -4.71572870 \end{cases} \quad (55)$$

Where after solving, we get the solutions:

$$\begin{cases} a = 1.65715376 \\ b = 1.65692730 \end{cases} \quad (56)$$

So that the algebraic function, according with elliptic sine prototype, describing our particular mechanical system, which contains information about the time-argument and eccentricity-argument of solutions family, is:

$$F_p(k, X) = -1.65715376\ \ (1 + 0.04827641\ \ \mathrm{k}^2)\mathrm{X} + 0.159981008\ \ \mathrm{k}^2\mathrm{X}^3 \quad (57)$$

And we could see that for:

$$k = k_{\max} = 1 \quad (58)$$

We get above function:

$$F_p = -1.7371552\ \ \mathrm{X} + 0.159981008\ \ \mathrm{X}^3 \quad (59)$$

In the graph following we have plotted, with continuous line, the initial algebraic function F_r (9), and with dashed line the F_p function, which is our funded function (59), in respect with prototype of elliptic function after the second order derivative, to show the very small difference (between the roots domain, where the function was defined).

Systems (44) are becoming:

$$\begin{cases} -x_0\alpha^2 - x_0\alpha^2 k^2 = -1.65715376 - 0.08000143\ k^2 \\ 2x_0\alpha^2 k^2 = 0.159981008\ k^2 \end{cases} \tag{60}$$

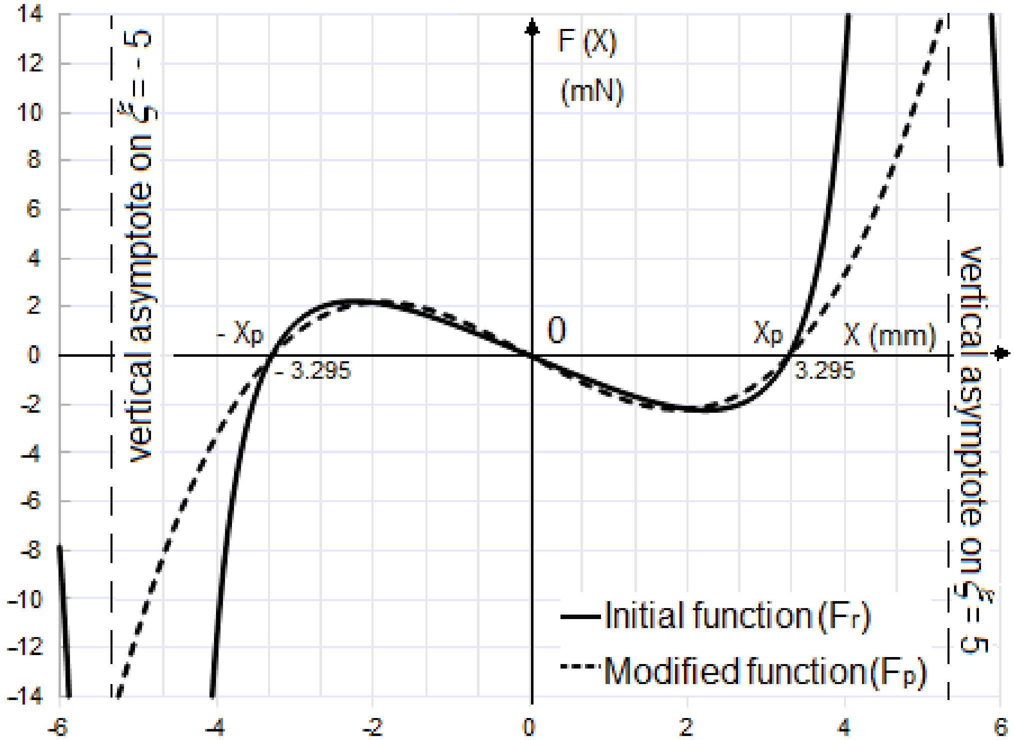

Fig. 4. Graph initial function (F_r), and prototype of elliptic function (F_p)

From where we find the argument form, for undetermined k (the above system is a linear dependent one):

$$\alpha = \frac{1.287}{\sqrt{x_0}} \tag{61}$$

Together (54) and (61), are compounding the form of our solution follow:

$$X(t) = x_0 sn\left[\frac{1.2873}{\sqrt{x_0}}t, \frac{4.551}{\sqrt{2x_0^2 - 1}}\right] \tag{62}$$

In the end, we have plotted funded solution formula for some representative value of initial displacement.

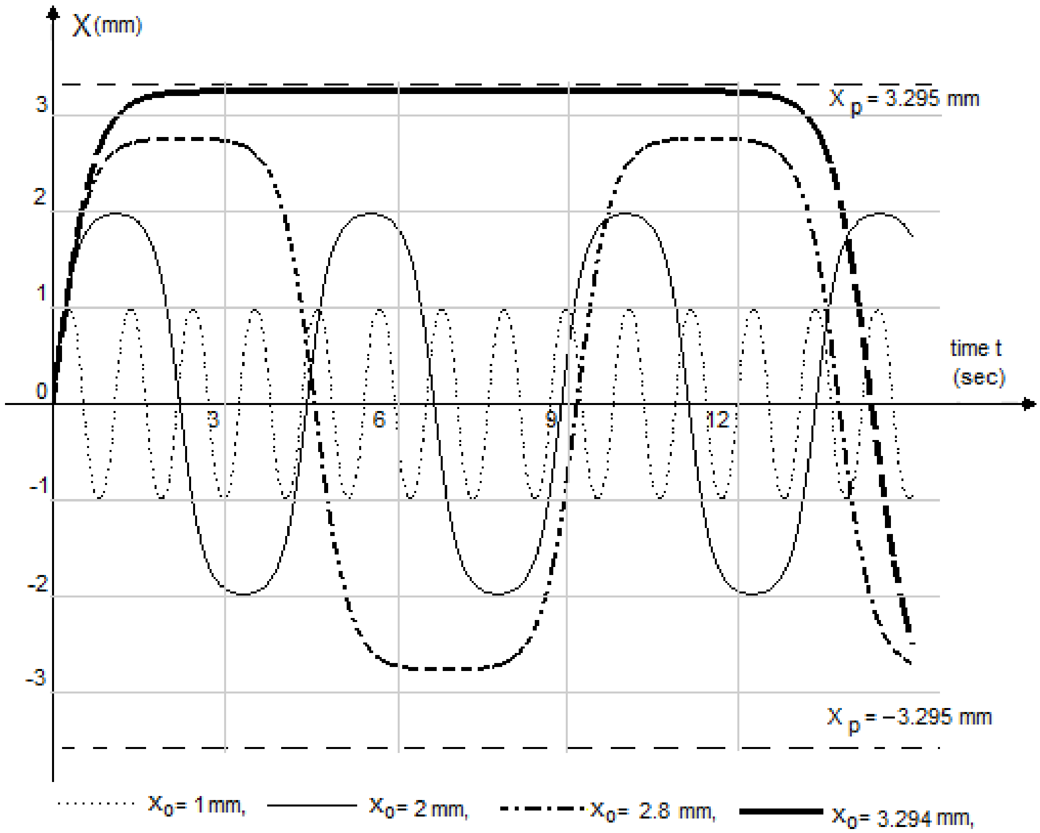

Fig. 5. Different type of oscillations, are produced in spring-magnet-mass system, depending on starting position displacement x_0, according with founded formula (62)

4 Conclusions

The founded formula (62) shows the dependence of time-argument and eccentricity-argument variation through the changes of initial starting position. If the maximum of eccentricity is reached, then the main oscillations stop for a while, and this halt of oscillation can be possible by the sudden decrease of speed motion, not by friction or loss of energy, but only by reaching maximum of nonlinearity.

We have proposed a particular form of elliptic sine function, then a second formal order derivative. Considering that the derivation of this function type generates a third order polynomial, we are looking for a polynomial prototype of this form, for the given function, firstly the algebraic way, finding an expression for a and b, and secondly equalizing inter graphics areas of these functions, obtaining another expression for a and b.

The founded elliptical polynomial function prototype, is in concordance with the initial given form and has the exact same algebraic solutions. Then by identification of coefficients, we have founded the time-argument and eccentricity-argument expression. Almost the same systems were analyzed also in the work of other authors [8,9] where the same movement equation depending on initial positions was described. The graphs show that initial impulse applied to the oscillator determines the shape of the law of motion, which is also experimentally proven by the authors.

Analytical solutions to this problem had been extensively studied by other authors, and some constructive ideas were taken from the specialty literature of Duffin oscillator subject, described in works [10,11].

Competing Interests

Authors have declared that no competing interests exist.

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