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A fixed point theorem for generalized weakly contractive mappings in b-metric spaces

Eliyas Zinab[1], Kidane Koyas[1] and Aynalem Girma[1,*]

[1] Department of Mathematics, College of Natural Sciences, Jimma University, Jimma, Ethiopia.; eliyaszinab7@gmail.com (E.Z); kidanekoyas@yahoo.com (K.K); aynalemgirma46@yahoo.com (A.G)

* Correspondence: aynalemgirma46@yahoo.com

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Abstract: In this paper we establish a fixed point theorem for generalized weakly contractive mappings in the setting of b-metric spaces and prove the existence and uniqueness of a fixed point for a self-mappings satisfying the established theorem. Our result extends and generalizes the result of Cho [1]. Finally, we provided an example in the support of our main result.

Keywords: Fixed point, generalized weak contractive mapping, b-metric space.

MSC: 26D10, 31A10, 26A33.

1. Introduction

In 1993, Czerwik [2] introduced the concept of b-metric spaces and proved the Banach contraction mapping principle in the setting of b-metric spaces. Afterwards, several research papers [3–8] were published on the existence of fixed point results for single-valued and multi-valued mappings in the setting of b-metric spaces. In 1997, Alber *et al.* [9] generalized Banach's contraction principle by introducing the concept of weakly contractive mappings and proved the existence of fixed points for weakly contractive and single valued mappings on Hilbert spaces.

Rhoades [10] proved that every weakly contractive mapping has a unique fixed point in complete metric spaces. Then, many authors obtained generalizations and extensions of the weakly contractive mappings.

In particular, Choudhury *et al.* [11] generalized fixed point results for weakly contractive mappings by using altering distance functions. Very recently, Cho [1] introduced the notion of generalized weakly contractive mappings in metric spaces and proved a fixed point theorem for generalized weakly contractive mappings defined on complete metric spaces.

Inspired and motivated by the results of Cho [1] the purpose of this paper is to establish a fixed point result for generalized weakly contractive mappings in the setting of b-metric spaces.

2. Preliminaries

In this section, we give basic definitions of concepts concerning a generalized weakly contractive mappings in the setting of b-metric spaces.

Definition 1. [2] Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric if and only if for all $x, y, z \in X$, the following conditions are satisfied:

(a) $d(x, y) = 0$ if and only if $x = y$;
(b) $d(x, y) = d(y, x)$;
(c) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces, since b-metric is metric when $s = 1$. But, in general, the converse is not true.

Example 1. [12] Let $X = R$ and $d : X \times X \to R^+$ be given by $d(x,y) = (x-y)^2$ for all $x, y \in X$, then d is a b-metric on X with $s = 2$ but it is not a metric on X, because for $x = 2, y = 4$ and $z = 6$, we have $d(2,6) \nleq 2[d(2,4) + d(4,6)]$, hence the triangle inequality for a metric does not hold.

Definition 2. A function $f \colon X \to R^+$, where X is b-metric space is called lower semicontinuous if for all $x \in X$ and $x_n \in X$ with $\lim_{n\to\infty} x_n = x$, we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

Definition 3. [6] Let X be a b-metric space and $\{x_n\}$ be a sequence in X, we say that

(a) x_n is b-converges to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$.
(b) x_n is a b-Cauchy sequence if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
(c) (X, d) is b-complete if every b-Cauchy sequence in X is b-convergent.

Definition 4. [1] Let X be a complete metric space with metric d, and $T \colon X \to X$. Also let $\varphi \colon X \to R^+$ be a lower semicontinuous function, then T is called a generalized weakly contractive mapping if it satisfies the following condition:

$$\psi(d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x,y,d,T,\varphi)) - \phi(l(x,y,d,T,\varphi))$$

where,

$$\begin{aligned} m(x,y,d,T,\varphi) &= max\,\{d(x,y) + \varphi(x) + \varphi(y), d(x,Tx) + \varphi(x) + \varphi(Tx), d(y,Ty) + \varphi(y) + \varphi(Ty), \\ &\quad \frac{1}{2}\left[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)\right]\Big\} \end{aligned}$$

and $l(x,y,d,T,\varphi) = \max\{d(x,y) + \varphi(x) + \varphi(y), d(y,Ty) + \varphi(y) + \varphi(Ty)\}$, for all $x, y \in X$, where $\psi \colon R^+ \to R^+$ is a continuous with $\psi(t) = 0$ if and only if $t = 0$ and $\phi \colon R^+ \to R^+$ is a lower semicontinuous function with $\phi(t) = 0$ if and only if $t = 0$.

Theorem 1. *[1] Let X be complete. If T is a generalized weakly contractive mapping, then there exists a unique $z \in X$ such that $z = Tz$ and $\varphi(z) = 0$.*

Lemma 1. *[12] Suppose (X, d) is a b-metric space and $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_n, x_{n+1}) \to 0.$$

If $\{x_n\}$ is not a b-Cauchy sequence, then there exists $\epsilon > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) \geq k$ such that for all positive intiger k, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$, $d(x_{m(k)}, x_{n(k-1)}) < \epsilon$ and

(a) $\epsilon \leq \lim_{n\to\infty} inf d(x_{m(k)}, x_{n(k)}) \leq \lim_{n\to\infty} sup d(x_{m(k)}, x_{n(k)}) \leq s\epsilon$.
(b) $\frac{\epsilon}{s} \leq \lim_{n\to\infty} inf d(x_{m(k)}, x_{n(k)}) \leq \lim_{n\to\infty} sup d(x_{m(k+1)}, x_{n(k)}) \leq s^2\epsilon$.
(c) $\frac{\epsilon}{s^2} \leq \lim_{n\to\infty} inf d(x_{m(k+1)}, x_{n(k)}) \leq \lim_{n\to\infty} sup d(x_{m(k)}, x_{n(k+1)}) \leq s^2\epsilon$.
(d) $\frac{\epsilon}{s^2} \leq \lim_{n\to\infty} inf d(x_{m(k+1)}, x_{n(k+1)}) \leq \lim_{n\to\infty} sup d(x_{m(k+1)}, x_{n(k+1)}) \leq s^3\epsilon$

holds.

3. Results and discussion

In this section, we introduce a generalized weakly contractive mappings in the setting of b-metric spaces and prove a fixed point result.

Definition 5. Let X be a b-metric space with metric d and parameter $s \geq 1$, $T : X \to X$, and let $\varphi : X \to R^+$ be a lower semicontinuous function, then T is called a generalized weakly contractive mapping if satisfies the following condition:

$$\psi(S^3 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)) \leq \psi(m(x,y,d,T,\varphi)) - \phi(l(x,y,d,T,\varphi)) \tag{1}$$

for all $x, y \in X$, where,

$$m(x,y,d,T,\varphi) = max\left\{d(x,y)+\varphi(x)+\varphi(y), d(x,Tx)+\varphi(x)+\varphi(Tx), d(y,Ty)+\varphi(y)+\varphi(Ty), \frac{1}{2s^2}\{d(x,Ty)+\varphi(x)+\varphi(Ty)+d(y,Tx)+\varphi(y)+\varphi(Tx)\}\right\} \quad (2)$$

and

$$l(x,y,d,T,\varphi) = max\{d(x,y)+\varphi(x)+\varphi(y), d(y,Ty)+\varphi(y)+\varphi(Ty)\}. \quad (3)$$

for all $x, y \in X$, where $\psi \colon R^+ \to R^+$ is a continuous with $\psi(t) = 0$ if and only if $t = 0$ and $\phi \colon R^+ \to R^+$ is a lower semicontinuous function with $\phi(t) = 0$ if and only if $t = 0$.

Theorem 2. *Let X be a complete b-metric space with metric d and $s \geq 1$. If T is a generalized weakly contractive mapping then T has a unique fixed point $u \in X$ such that $u = Tu$ and $\varphi(u) = 0$.*

Proof. Let $x_0 \in X$ be fixed and define a sequence $\{x_n\}$ by $x_1 = Tx_0$, $x_2 = Tx_1, \ldots, x_{n+1} = Tx_n$ for all $n = 0, 1, 2, \ldots$. If $x_n = x_{n+1}$ for some n, $x_n = x_{n+1} = Tx_n$, x_n is fixed point of T.

Assume $x_n \neq x_{n+1}$ for all $n = 0, 1, 2, \ldots$. From (2) by using $x = x_{n-1}$ and $y = x_n$, we have $m(x_{n-1},x_n,d,T,\varphi) = \max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_{n-1},Tx_{n-1})+\varphi(x_{n-1})+\varphi(Tx_{n-1}), d(x_n,Tx_n)+\varphi(x_n)+\varphi(Tx_n), \frac{1}{2s^2}\{d(x_{n-1},Tx_n)+\varphi(x_{n-1})+\varphi(Tx_n)+d(x_n,Tx_{n-1})+\varphi(x_n)+\varphi(Tx_{n-1})\}\} = \max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}), \frac{1}{2s^2}\{d(x_{n-1},x_{n+1})+\varphi(x_{n-1})+\varphi(x_{n+1})+d(x_n,x_n)+\varphi(x_n)+\varphi(x_n)\}\}$.

Since $\frac{1}{2s^2}\{d(x_{n-1},x_{n+1})+\varphi(x_{n-1})+\varphi(x_{n+1})+d(x_n,x_n)+\varphi(x_n)+\varphi(x_n)\} \leq \frac{1}{2s^2}\{sd(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)+sd(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\} \leq \frac{1}{2s^2}\{s[d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)+d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})]\} = \frac{1}{2s}\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)+d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\} \leq \frac{1}{2}\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)+d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\} \leq max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}$.

So, we obtain

$$m(x_{n-1},x_n,d,T,\varphi) = max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}. \quad (4)$$

Similarly from (3)

$$l(x_{n-1},x_n,d,T,\varphi) = max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}. \quad (5)$$

Then (1) becomes

$$\psi(s^3d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}) \leq \psi(m(x_{n-1},x_n,d,T,\varphi)) - \phi(l(x_{n-1},x_n,d,T,\varphi)). \quad (6)$$

Now, if $d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n) \leq d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})$, for some positive integer n then (6) becomes $\psi(s^3d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}) \leq \psi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})) - \phi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))$.

It follows $\psi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})) \leq \psi(s^3d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}) \leq \psi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})) - \phi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))\psi(d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}))$, which is a contradiction. Thus,

$$d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n) > d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}). \quad (7)$$

From (4), (5) and (7), we obtain

$$m(x_{n-1},x_n,d,T,\varphi) = l(x_{n-1},x_n,d,T,\varphi) = d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), \quad (8)$$

So (6) becomes:

$$\psi(s^3d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1}) \leq \psi(m(x_{n-1},x_n,d,T,\varphi)) - \phi(l(x_{n-1},x_n,d,T,\varphi)). \quad (9)$$

From (7), the sequence $(d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}))$ is decreasing and bounded below. Hence $d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) \to r$ as $n \to \infty$ for some $r \geq 0$. Assume $r > 0$ and letting $n \to \infty$ in (9) and using the continuity of ψ and the lower semicontinuity of ϕ, we have

$$\begin{aligned} \psi(s^3 r) &\leq \psi(r) - \lim_{n\to\infty} inf \phi(d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)) \\ &\leq \psi(r) - \lim_{n\to\infty} \phi(d(x_{n-1}, x_n) + \varphi(x_{n-1}) + \varphi(x_n)) \\ &= \psi(r) - \phi(r). \end{aligned}$$

It follows that $\psi(r) \leq \psi(s^3 r) \leq \psi(r) - \phi(r) < \psi(r)$, which is a contradiction, hence we have $r = 0$ and consequently, $\lim_{n\to\infty}[d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1})] = 0$. Implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{10}$$

$$\lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} \varphi(x_{n+1}) = 0. \tag{11}$$

Now, we prove that the sequence $\{x_n\}$ is a b-Cauchy sequence. If $\{x_n\}$ is not a b-Cauchy sequence, then by Lemma 1 there exists $\epsilon > 0$ and sequences of positive integers $m(k)$ and $n(k)$ such that for all positive integer k, $n(k) > m(k) \geq k$, $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$ and $d(x_{m(k)}, x_{n(k-1)}) < \epsilon$ and conditions from (a)-(d) of 1 hold.

From (2) and by setting $x = x_{m(k)}$ and $y = x_{n(k)}$ we have:
$m(x_{m(k)}, x_{n(k)}, d, T, \varphi) = \max\{(d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}), d(x_{m(k)}, Tx_{m(k)}) + \varphi(x_{m(k)}) + \varphi(Tx_{m(k)}), d(x_{n(k)}, Tx_{n(k)}) + \varphi(x_{n(k)}) + \varphi(Tx_{n(k)}), \frac{1}{2s^2}\{d(x_{m(k)}, Tx_{n(k)}) + \varphi(x_{m(k)}) + \varphi(Tx_{n(k)}) + d(x_{n(k)}, Tx_{m(k)}) + \varphi(x_{n(k)}) + \varphi(Tx_{m(k)})\}\} = \max\{d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}) + \varphi(x_{m(k)}) + \varphi(x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}) + \varphi(x_{n(k)}) + \varphi(x_{n(k)+1}), \frac{1}{2S^2}\{d(x_{m(k)}, x_{n(k)+1}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1}) + \varphi(x_{n(k)}) + \varphi(x_{m(k)+1})\}\}$.

Taking the limit as $k \to \infty$ and using (10), (11) and Lemma 1, we have

$$\begin{aligned} &\lim_{k\to\infty} m(x_{m(k)}, x_{n(k)}, d, T, \varphi) = \max \lim_{k\to\infty} \{d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}) + \varphi(x_{m(k)}) \\ &+\varphi(x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}) + \varphi(x_{n(k)}) + \varphi(x_{n(k)+1}), \frac{1}{2S^2}\{d(x_{m(k)}, x_{n(k)+1}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)+1}) \\ &+d(x_{n(k)}, x_{m(k)+1}) + \varphi(x_{n(k)}) + \varphi(x_{m(k)+1})\}\} \leq max\{s\epsilon, 0, 0, \frac{1}{2s^2}(s^2\epsilon + s^2\epsilon)\} = s\epsilon. \end{aligned} \tag{12}$$

Similarly from (3), we have
$l(x_{m(k)}, x_{n(k)}, d, T, \varphi) = max\{d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}), d(x_{n(k)}, Tx_{n(k)}) + \varphi(x_{n(k)}) + \varphi(Tx_{n(k)})\} = max\{d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}), d(x_{n(k)}, x_{n(k)+1}) + \varphi(x_{n(k)}) + \varphi(x_{n(k)+1})\}$

$$\begin{aligned} &\lim_{k\to\infty} l(x_{m(k)}, x_{n(k)}, d, T, \varphi) = \lim_{k\to\infty} max\{d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}), d(x_{n(k)}, x_{n(k)+1}) + \varphi(x_{n(k)}) \\ &+\varphi(x_{n(k)+1})\} \leq \max\{s\epsilon, 0\} = s\epsilon. \end{aligned} \tag{13}$$

Now from (1), we have
$\psi(s^3 d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)) = \psi(s^3 d(x_{m(k)+1}, x_{n(k)+1}) + \varphi(x_{m(k)+1}) + \varphi(x_{n(k)+1})) \leq \psi(m(x_{m(k)}, x_{n(k)}, d, T, \varphi)) - \phi(l(x_{m(k)}, x_{n(k)}, d, T, \varphi))$.

Letting $k \to \infty$, using (11), (12), (13), applying the continuity of ψ and lower semicontinuity of ϕ, we have,

$$\lim_{k\to\infty} \psi(s^3 d(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(s\epsilon) - \phi(s\epsilon).$$

This implies that

$$\psi(s\epsilon) = \psi(s^3 \frac{\varepsilon}{s^2}) \leq \psi(s^3 \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1})) \leq \psi(s\varepsilon) - \phi(s\epsilon) < \psi(s\epsilon),$$

which is a contradiction. Therefore $\{x_n\}$ is a b-Cauchy sequence. Now since $\{x_n\}$ is a b-Cauchy and X is b-complete we have,

$$\lim_{n\to\infty} x_n = u \in X.$$

Since φ is lower semicontinuous,

$$\varphi(u) \leq \lim_{n\to\infty} inf \varphi(x_n) \leq \lim_{n\to\infty} \varphi(x_n) = 0,$$

which implies

$$\varphi(u) = 0. \tag{14}$$

Now from (2) by putting $x = x_n$ and $y = u$, we have
$m(x_n, u, d, T, \varphi) = max\{d(x_n, u) + \varphi(x_n) + \varphi(u), d(x_n, Tx_n) + \varphi(x_n) + \varphi(Tx_n), d(u, Tu) + \varphi(u) + \varphi(Tu), \frac{1}{2s^2}\{d(x_n, Tu) + \varphi(x_n) + \varphi(Tu) + d(u, Tx_n) + \varphi(u) + \varphi(Tx_n)\}\} = max\{d(x_n, u) + \varphi(x_n) + \varphi(u), d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(u, Tu) + \varphi(u) + \varphi(Tu), \frac{1}{2s^2}\{d(x_n, Tu) + \varphi(x_n) + \varphi(Tu) + d(u, x_{n+1}) + \varphi(u) + \varphi(x_{n+1})\}\}$.

Applying the limit as $n \to \infty$ and using (10), (11) and(14) we have

$$\lim_{n\to\infty} m(x_n, u, d, T, \varphi) = d(u, Tu) + \varphi(Tu). \tag{15}$$

Similarly

$$\lim_{n\to\infty} l(x_n, u, d, T, \varphi) = d(u, Tu) + \varphi(Tu). \tag{16}$$

Then using (1), we have

$$\begin{aligned} \psi(s^3 d(Tx_n, Tu) + \varphi(Tx_n) + \varphi(Tu)) &= \psi(s^3 d(x_{n+1}, Tu) + \varphi(x_{n+1}) + \varphi(Tu)) \\ &\leq \psi(m(x_n, u, d, T, \varphi)) - \phi(l(x_n, u, d, T, \varphi)). \end{aligned}$$

Letting $n \to \infty$, using (14),(15), (16) and by using the continuity of ψ and lower semicontinuity of ϕ, we have

$$\begin{aligned} \psi(s^3 d(x_{n+1}, Tu) + \varphi(x_{n+1}) + \varphi(Tu)) &= \psi(s^3 d(u, Tu) + \varphi(Tu)) \\ &\leq \psi(m(x_n, u, d, T, \varphi)) - \phi(l(x_n, u, d, T, \varphi)) \\ &= \psi(d(u, Tu) + \varphi(Tu)) - \phi(d(u, Tu) + \varphi(Tu)). \end{aligned}$$

This implies

$$\begin{aligned} \psi(d(u, Tu) + \varphi(Tu)) &\leq \psi(s^3 d(u, Tu) + \varphi(Tu)) \\ &\leq \psi(d(u, Tu) + \varphi(Tu)) - \phi(d(u, Tu) + \varphi(Tu)). \end{aligned}$$

This holds if and only if, $\phi(d(u, Tu) + \varphi(Tu)) = 0$ and then from the property of ϕ we have,

$$d(u, Tu) + \varphi(Tu) = 0.$$

Hence, $d(u, Tu) = 0$ so that u = Tu and $\varphi(Tu) = 0$. Since $u = Tu$ this implies $\varphi(u) = 0$.
Therefore u is fixed point of T.

Uniqueness

Suppose v is another fixed point of T. Then $Tv = v$ and $\varphi(v) = 0$.
By (1) with $x = u$ and $y = v$

$$\psi(s^3 d(Tu, Tv) + \varphi(Tu) + \varphi(Tv)) = \psi(s^3 d(u, v)) \leq \psi(m(Tu, Tv, d, T, \varphi)) - \phi(l(Tu, Tv, d, T, \varphi)).$$

From (2) we have
$m(Tu, Tv, d, T, \varphi) = max\{(d(Tu, Tv) + \varphi(Tu) + \varphi(Tv), d(Tu, T^2u) + \varphi(Tu) + \varphi(T^2u), d(Tv, T^2v) + \varphi(Tv) + \varphi(T^2v), \frac{1}{2s^2}\{d(Tu, T^2v) + \varphi(Tu) + \varphi(T^2v) + d(Tv, T^2u) + \varphi(Tv) + \varphi(T^2u)\}\} = max\{(d(u, v) + \varphi(u) + \varphi(v), d(u, u) + \varphi(u) + \varphi(u), d(v, v) + \varphi(v) + \varphi(v), \frac{1}{2s^2}\{d(u, v) + \varphi(u) + \varphi(v) + d(v, u) + \varphi(v) + \varphi(u)\}\} = d(u, v)$.

Similarly from (3), we have

$$\begin{aligned} l(Tu,Tv,d,T,\varphi) &= max\{d(Tu,Tv)+\varphi(Tu)+\varphi(Tv), d(Tv,T^2v)+\varphi(Tv)+\varphi(T^2v) \\ &= d(u,v)+\varphi(u)+\varphi(v), d(v,v)+\varphi(v)+\varphi(v)\} = d(u,v). \end{aligned}$$

Then using (1) and the continuity of ψ, we have

$$\psi(d(u,v)) \quad \leq \quad \psi(s^3 d(u,v)) \leq \psi(d(u,v)) - \phi(d(u,v)).$$

This holds if $\phi(d(u,v)) = 0$ and then we have $d(u,v) = 0$. Hence $u = v$. Therefore, T has a unique fixed point. □

Example 2. Let $X = [0,1]$ and $d(x,y) = (x-y)^2$. Then (X,d) is b-metric space with $s = 2$. Define $T : X \to X$, $\varphi : X \to R^+$ *and* $\psi, \phi : R^+ \to R^+$ by $T(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{4} \\ \frac{1}{16} & \text{if } x \in (\frac{1}{4}, 1] \end{cases}$, $\psi(t) = \frac{5t}{4}$, $\phi(t) = \begin{cases} \frac{t}{8} & \text{if } t \leq 3 \\ \frac{t}{4} & \text{if } t > 3 \end{cases}$ and $\varphi(t) = \begin{cases} 2t & \text{if } t > 1 \\ t & \text{if } 0 \leq t \leq 1 \end{cases}$. Now we verify condition (1).

Case I: If $x, y \in [0, \frac{1}{4}]$ and $x \geq y$. Then $\psi[s^3 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)] = \psi[2^3(Tx-Ty)^2 + \varphi(Tx) + \varphi(Ty)] = \psi[8(0) + \varphi(0) + \varphi(0)] = 0$.

Also $d(x,y) + \varphi(x) + \varphi(y) = (x-y)^2 + \varphi(x) + \varphi(y) = (x-y)^2 + x + y$,
$d(x,Tx) + \varphi(x) + \varphi(Tx) = (x-Tx)^2 + \varphi(x) + \varphi(Tx) = x^2 + x$,
$d(y,Ty) + \varphi(y) + \varphi(Ty) = (y-Ty)^2 + \varphi(y) + \varphi(Ty) = y^2 + y$,
$\frac{1}{s^2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)] = \frac{1}{8}(x^2 + x + y^2 + y)$.

And $m(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, x^2 + x, y^2 + y, \frac{1}{8}(x^2 + x + y^2 + y)\}$,
$\frac{1}{8}(x^2 + x + y^2 + y) \leq max\{x^2 + x, y^2 + y\} = x^2 + x$,
$m(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, x^2 + x\}$
and $l(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, y^2 + y\}$.

But $(x-y)^2 + x + y \geq x + y \geq y + y \geq y^2 + y$, so , $l(x,y,d,T,\varphi) = (x-y)^2 + x + y$ and $\phi[l(x,y,d,T,\varphi)] = \frac{1}{8}[(x-y)^2 + x + y]$.

If $m(x,y,d,T,\varphi) = (x-y)^2 + x + y$ then (1) becomes $0 \leq \frac{5}{4}[(x-y)^2 + x + y] - \frac{1}{8}[(x-y)^2 + x + y] = \frac{9}{8}[(x-y)^2 + x + y]$ and if $m(x,y,d,T,\varphi) = x^2 + x$ then by (1), we have $\frac{5}{4}(x^2 + x) - \frac{1}{8}[(x-y)^2 + x + y] \geq \frac{5}{4}(x^2 + x) - \frac{1}{8}(x^2 + x) = \frac{9}{8}(x^2 + x) \geq 0$.

Let $x, y \in [0, \frac{1}{4}]$ and $x < y$. Then $m(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, x^2 + x, y^2 + y, \frac{1}{8}(x^2 + x + y^2 + y)\}$, $\frac{1}{8}(x^2 + x + y^2 + y) \leq max\{x^2 + x, y^2 + y\} = y^2 + y$, implies $m(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, y^2 + y\}$.

Similarly $l(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, y^2 + y\}$. Now, if $m(x,y,d,T,\varphi) = (x-y)^2 + x + y = l(x,y,d,T,\varphi)$, (1) becomes $0 \leq \frac{9}{8}[(x-y)^2 + x + y]$. If $m(x,y,d,T,\varphi) = y^2 + y = l(x,y,d,T,\varphi)$, (1) becomes $0 \leq \frac{9}{8}(y^2 + y)$.

Case II: If $x \in [0, \frac{1}{4}], y \in (\frac{1}{4}, 1]$. This implies $x < y$. Then $\psi[s^3 d(Tx,Ty) + \varphi(Tx) + \varphi(Ty)] = \psi[2^3(Tx-Ty)^2 + \varphi(Tx) + \varphi(Ty)] = \frac{5}{4}[8(0 - \frac{1}{16})^2 + \varphi(0) + \varphi(\frac{1}{16})] = \frac{5}{4}(\frac{1}{32} + \frac{1}{16}) = \frac{15}{128}$.

Also $d(x,y) + \varphi(x) + \varphi(y) = (x-y)^2 + \varphi(x) + \varphi(y) = (x-y)^2 + x + y$,
$d(x,Tx) + \varphi(x) + \varphi(Tx) = (x-Tx)^2 + \varphi(x) + \varphi(Tx) = x^2 + x$,
$d(y,Ty) + \varphi(y) + \varphi(Ty) = (y - \frac{1}{16})^2 + y + \frac{1}{16} = y^2 + \frac{7y}{8} + \frac{17}{256}$,
$\frac{1}{s^2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)] = \frac{1}{8}[x^2 + \frac{7x}{8} + y^2 + y + \frac{17}{256}]$.

And $m(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, x^2 + x, y^2 + \frac{7y}{8} + \frac{17}{256}, \frac{1}{8}[x^2 + \frac{7x}{8} + y^2 + y + \frac{17}{256}]\}$,
$\frac{1}{8}[x^2 + \frac{7x}{8} + y^2 + y + \frac{17}{256}] = \frac{1}{4}[\frac{x^2}{2} + \frac{7x}{16} + \frac{y^2}{2} + \frac{y}{2} + \frac{17}{512}] \leq \frac{1}{4}[x^2 + x + y^2 + \frac{7y}{8} + \frac{17}{256}] \leq max\{x^2 + x, y^2 + \frac{7y}{8} + \frac{17}{256}\} = y^2 + \frac{7y}{8} + \frac{17}{256}$.

Therefore $m(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, y^2 + \frac{7y}{8} + \frac{17}{256}\}$. Similarly, $l(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, y^2 + \frac{7y}{8} + \frac{17}{256}\}$.

Now if, $m(x,y,d,T,\varphi) = (x-y)^2 + x + y = l(x,y,d,T,\varphi)$, then $\psi[m(x,y,d,T,\varphi)] = \frac{5}{4}[(x-y)^2 + x + y]$ and $\phi[m(x,y,d,T,\varphi)] = \frac{1}{8}[(x-y)^2 + x + y]$. So (1) becomes $\frac{15}{128} \leq \frac{9}{8}[(x-y)^2 + x + y]$.

Also if, $m(x,y,d,T,\varphi) = y^2 + \frac{7y}{8} + \frac{17}{256} = l(x,y,d,T,\varphi)$, then $\psi[m(x,y,d,T,\varphi)] = \frac{5}{4}[y^2 + \frac{7y}{8} + \frac{17}{256}]$ and $\phi[m(x,y,d,T,\varphi)] = \frac{1}{8}[y^2 + \frac{7y}{8} + \frac{17}{256}]$. So (1) becomes $\frac{15}{128} \leq \frac{9}{8}[y^2 + \frac{7y}{8} + \frac{17}{256}]$.

Case III: If $x, y \in (\frac{1}{4}, 1]$ and $x \geq y$. Then $\psi[s^3 d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)] = \psi[8(\frac{1}{16} - \frac{1}{16})^2 + \frac{1}{16} + \frac{1}{16}] = \psi[\frac{1}{8}] = \frac{5}{32}$.

Also $d(x,y) + \varphi(x) + \varphi(y) = (x-y)^2 + x + y$, $d(x,Tx) + \varphi(x) + \varphi(Tx) = (x - \frac{1}{16})^2 + x + \frac{1}{16}$, $d(y,Ty) + \varphi(y) + \varphi(Ty) = (y - \frac{1}{16})^2 + y + \frac{1}{16}$,
$\frac{1}{s^2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)] = \frac{1}{8}[(x - \frac{1}{16})^2 + x + \frac{1}{16} + (y - \frac{1}{16})^2 + y + \frac{1}{16})] \leq max[(x - \frac{1}{16})^2 + x + \frac{1}{16}, (y - \frac{1}{16})^2 + y + \frac{1}{16}] = (x - \frac{1}{16})^2 + x + \frac{1}{16}$.

This implies that $m(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, (x - \frac{1}{16})^2 + x + \frac{1}{16}\}$ and $l(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, (y - \frac{1}{16})^2 + y + \frac{1}{16}\}$. But $(y - \frac{1}{16})^2 + y + \frac{1}{16} = (\frac{1}{16} - y)^2 + y + \frac{1}{16} < (x-y)^2 + y + x$. So $l(x,y,d,T,\varphi) = (x-y)^2 + x + y$.

Now if, $m(x,y,d,T,\varphi) = (x-y)^2 + x + y$ (1) becomes, $\frac{5}{32} \leq \frac{5}{4}[(x-y)^2 + x + y] - \frac{1}{8}[(x-y)^2 + x + y] = \frac{9}{8}[(x-y)^2 + x + y]$.

If $m(x,y,d,T,\varphi) = (x - \frac{1}{16})^2 + x + \frac{1}{16}$ then we have, $\frac{5}{4}[(x - \frac{1}{16})^2 + x + \frac{1}{16}] - \frac{1}{8}[(x-y)^2 + x + y] > \frac{5}{4}[(x - \frac{1}{16})^2 + x + \frac{1}{16}] - \frac{1}{8}[(x - \frac{1}{16})^2 + x + \frac{1}{16}] = \frac{9}{8}[(x - \frac{1}{16})^2 + x + \frac{1}{16}] > \frac{5}{32}$.

Let $x, y \in (\frac{1}{4}, 1]$ and $x < y$. Then $m(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, (y - \frac{1}{16})^2 + y + \frac{1}{16}\} = l(x,y,d,T,\varphi)$.

Now if $m(x,y,d,T,\varphi) = l(x,y,d,T,\varphi) = (x-y)^2 + x + y$, then (1) becomes $\frac{5}{32} \leq \frac{5}{4}[(x-y)^2 + x + y] - \frac{1}{8}[(x-y)^2 + x + y] = \frac{9}{8}[(x-y)^2 + x + y]$.

If $m(x,y,d,T,\varphi) = l(x,y,d,T,\varphi) = (y - \frac{1}{16})^2 + y + \frac{1}{16}$, (1) becomes $\frac{5}{32} \leq \frac{5}{4}[(y - \frac{1}{16})^2 + y + \frac{1}{16}] - \frac{1}{8}[(y - \frac{1}{16})^2 + y + \frac{1}{16}] = \frac{9}{8}[(y - \frac{1}{16})^2 + y + \frac{1}{16}]$.

Case IV: If $x \in (\frac{1}{4}, 1], y \in [0, \frac{1}{4}]$. This implies $x > y$. Then $\psi[s^3 d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)] = \psi[2^3(Tx - Ty)^2 + \varphi(Tx) + \varphi(Ty)] = \frac{5}{4}[8(\frac{1}{16} - 0)^2 + \varphi(\frac{1}{16}) + +\varphi(0)] = \frac{5}{4}(\frac{1}{32} + \frac{1}{16}) = \frac{15}{128}$.

Also $d(x,y) + \varphi(x) + \varphi(y) = (x-y)^2 + x + y$,
$d(x,Tx) + \varphi(x) + \varphi(Tx) = x^2 + \frac{7x}{8} + \frac{17}{256}$,
$d(y,Ty) + \varphi(y) + \varphi(Ty) = y^2 + y$,
$\frac{1}{s^2}[d(x,Ty) + \varphi(x) + \varphi(Ty) + d(y,Tx) + \varphi(y) + \varphi(Tx)] = \frac{1}{8}[x^2 + x + y^2 + \frac{7y}{8} + \frac{17}{256}]$.

And $m(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, x^2 + \frac{7x}{8} + \frac{17}{256}, y^2 + y, \frac{1}{8}[x^2 + x + y^2 + \frac{7y}{8} + \frac{17}{256}]\}$,
$m(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, x^2 + \frac{7x}{8} + \frac{17}{256}\}$. Similarly, $l(x,y,d,T,\varphi) = max\{(x-y)^2 + x + y, y^2 + y\} = (x-y)^2 + x + y$.

Now if, $m(x,y,d,T,\varphi) = (x-y)^2 + x + y = l(x,y,d,T,\varphi)$, then (1) becomes $\frac{15}{128} \leq \frac{9}{8}[(x-y)^2 + x + y]$. If, $m(x,y,d,T,\varphi) = x^2 + \frac{7x}{8} + \frac{17}{256}$, we have $\frac{5}{4}[x^2 + \frac{7x}{8} + \frac{17}{256}] - \frac{1}{8}[(x-y)^2 + x + y] > \frac{5}{4}[x^2 + \frac{7x}{8} + \frac{17}{256}] - \frac{1}{8}[x^2 + \frac{7x}{8} + \frac{17}{256}] = \frac{9}{8}[x^2 + \frac{7x}{8} + \frac{17}{256}] > \frac{15}{128}$.

Then (1) becomes $\frac{15}{128} \leq \frac{9}{8}[x^2 + \frac{7x}{8} + \frac{17}{256}]$. Thus all the condition of Theorem (2) are satisfied and 0 is the unique fixed point of T.

Remark 1. If we take s=1 in Theorem (2) we get the result of Cho [1]. Hence Our result generalizes the result of Cho [1] and related results in the literature.

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