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Positive Solutions of Sturm-Liouville Problems for Second Order Singular and Impulsive Differential Equations

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Author's contribution

This whole work was carried out by the author YH.

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ABSTRACT

In this paper, we study the positive solutions of nonlinear singular two-point boundary value problems for second-order impulsive differential equations. The existence of positive solutions is established by using the fixed point theorem in cones.

Keywords: Positive solution; singular two-point boundary value problem; second-order impulsive differential equations; fixed point theorem.

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1. INTRODUCTION

Impulsive and singular differential equations play a very important role in modern applied mathematics due to their deep physical background and broad application. In this paper, we consider the existence of positive solutions of

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$$\begin{cases} -Lu = g(x,u), & x \in I', \\ -\Delta(pu')|_{x=x_k} = I_k(u(x_k)), & k = 1, 2, \cdots, m, \\ R_1(u) = \alpha_1 u(0) + \beta_1 u'(0) = 0, \\ R_2(u) = \alpha_2 u(1) + \beta_2 u'(1) = 0, \end{cases} \tag{1.1}$$

here $Lu = (p(x)u')' + q(x)u$ is sturm-liouville operator, $I = [0,1]$, $I' = I \setminus \{x_1, x_2, \cdots, x_m\}$ and $0 < x_1 < x_2 < \cdots < x_m < 1$ are given

$R^+ = [0, +\infty)$, $g \in C(I \times R^+, R^+)$, $I_k \in C(R^+, R^+)$, $\Delta(pu')|_{x=x_k} = p(x_k)u'(x_k^+) - p(x_k)u'(x_k^-)$, $u'(x_k^+)$ (respectively $u'(x_k^-)$) denotes the right limit (respectively left limit) of $u'(x)$ at $x = x_k$, $g(x,u)$ may be singular at $u = 0$.

Throughout this paper, we always suppose that

(S_1) $p(x) \in C^1([0,1], R)$, $p(x) > 0$, $q(x) \in C([0,1], R)$, $q(x) \le 0$, $\alpha_1, \alpha_2, \beta_2 \ge 0$, $\beta_1 \le 0$, $\alpha_1^2 + \beta_1^2 > 0$, $\alpha_2^2 + \beta_2^2 > 0$.

In recent years, boundary problems of second-order differential equations with impulses have been studied extensively in the literature (see for instance [1-9] and their references). In [1], Lin and Jiang studied the second-order impulsive differential equation with no singularity and obtained two positive solutions by using the fixed point index theorem in cones. However they did not consider the case when the function is singular. Motivated by the work mentioned above, we study the positive solutions of nonlinear singular two-point boundary value problems for second order impulsive differential equations (1.1) in this paper. Our argument is based on the fixed point theorem in cones.

Moreover, for the simplicity in the following discussion, we introduce the following hypotheses.

(H_1): There exists an $\varepsilon_0 > 0$ such that $g(x,u)$ and $I_k(u)$ are non increasing in $u \le \varepsilon_0$,for each fixed $x \in [0,1]$

(H_2): For each fixed $0 < \theta \le \varepsilon_0$

$$0 < \int_0^1 g(y, (\frac{m(y)}{m(1)} \frac{n(y)}{n(0)})\theta) dy < \infty$$

(see section2)

(H_3): $\varphi_1(x)$ is the eigenfunction related to the smallest eigenvalue λ_1 of the eigenvalue problem $-L\varphi = \lambda\varphi$, $R_1(\varphi) = R_2(\varphi) = 0$.

$$(H_4):\ g^{\infty}+\frac{\sum_{k=1}^{m} I^{\infty}(k)\varphi_1(x_k)}{\int_0^1 (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\varphi_1(x)dx}<\lambda_1,$$

where $g^{\infty}=\lim_{u\to+\infty}\sup\max_{x\in[0,1]}\frac{g(x,u)}{u},\quad I^{\infty}(k)=\lim_{u\to+\infty}\sup\frac{I_k(u)}{u},$

Theorem 1. Assume that $(H_1)-(H_4)$ are satisfied. Then problem (1.1) has at least one positive solution u .Moreover, there exists a $\theta^{*}>0$ such that

$$u(x)\ge\theta^{*}(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)}),\qquad x\in[0,1].$$

2. PRELIMINARY

In order to define the solution of (1.1) we shall consider the following space.

$PC'(I,R)=\{u\in C(I,R);u'|_{(x_k,x_{k+1})}\in C(x_k,x_{k+1}),u'(x_k^-)=u'(x_k),\exists\, u'(x_k^+),k=1,2,\cdots,m\}$ with the norm $\|u\|_{PC'}=\max\{\|u\|,\|u'\|\}$, here $\|u\|=\sup_{x\in[0,1]}|u(x)|$, $\|u'\|=\sup_{x\in[0,1]}|u'(x)|$. Then $PC'(I,R)$ is a Banach space.

Definition 2.1: A function u∈ $PC'(I,R)\cap C^2(I',R)$ is a solution of (1.1) if it satisfies the differential equation

$$Lu+g(x,u)=0,\quad x\in I'$$

and the function u satisfies conditions $\Delta(pu')|_{x=x_k}=-I_k(u(x_k))$ and $R_1(u)=R_2(u)=0.$

Let $Q=I\times I$ and $Q_1=\{(x,y)\in Q\,|\,0\le x\le y\le 1\},Q_2=\{(x,y)\in Q\,|\,0\le y\le x\le 1\}.$ Let $G(x,y)$ is the Green's function of the boundary value problem

$$-Lu=0,\ R_1(u)=R_2(u)=0.$$

Following from $[6]$, $G(x,y)$ can be written by

$$G(x,y):=\begin{cases}\dfrac{m(x)n(y)}{\omega},(x,y)\in Q_1,\\ \dfrac{m(y)n(x)}{\omega},(x,y)\in Q_2.\end{cases}\tag{2.1}$$

Lemma 2.1 [10]: Suppose that (S_1) holds, then the Green's function $G(x,y)$, defined by (2.1), possesses the following properties:

(i): $m(x)\in C^2(I,R)$ is increasing and $m(x)>0,\ x\in(0,1]$.

(ii): $n(x)\in C^2(I,R)$ is decreasing and $n(x)>0,\ x\in[0,1)$.

(iii): $(Lm)(x)\equiv 0,\ m(0)=-\beta_1,\ m'(0)=\alpha_1$.

(iv): $(Ln)(x)\equiv 0,\ n(1)=\beta_2,\ n'(1)=-\alpha_2$.

(v): ω is a positive constant. Moreover, $p(x)(m'(x)n(x)-m(x)n'(x))\equiv\omega$.

(vi): $G(x,y)$ is continuous and symmetrical over Q.

(vii): $G(x,y)$ has continuously partial derivative over Q_1, Q_2.

(viii): For each fixed $y\in I$, $G(x,y)$ satisfies $LG(x,y)=0$ for $x\neq y, x\in I$.
Moreover, $R_1(G)=R_2(G)=0$ for $y\in(0,1)$.

(viiii): G'_x has discontinuous point of the first kind at $x=y$ and

$$G'_x,\ G'_x(y+0,y)-G'_x(y-0,y)=-\frac{1}{p(y)},\ y\in(0,1).$$

Consider the linear Sturm-Liouvile problem

$$-(Lu)(x)=\lambda u(x),\quad R_1(u)=R_2(u)=0.$$

By the Sturm-Liouvile theory of ordinary differential equations, we know that there exists an eigenfunction $\varphi_1(x)$ with respect to the first eigenvalue $\lambda_1>0$ such that $\varphi_1(x)>0$ for $x\in(0,1)$.

Following from Lemma 2.1, it is easy to see that

$$\left(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)}\right)\frac{m(y)n(y)}{\omega}\le G(x,y)\le G(y,y)=\frac{m(y)n(y)}{\omega},\ (x,y)\in[0,1]\times[0,1]. \tag{2.2}$$

Lemma 2.2 [9]: If u is a solution of the equation

$$u(x)=\int_0^1 G(x,y)g(y,u(y))dy+\sum_{k=1}^{m}G(x,x_k)I_k(u(x_k)),\ x\in I. \tag{2.3}$$

then u is a solution of (1.1).

In fact by using inequalities (2.2), we have that

$$\|u\| \le \int_0^1 G(y,y)g(y,u(y))dy + \sum_{k=1}^{m} G(x_k,x_k)I_k(u(x_k))$$

and

$$u(x) \ge (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\int_0^1 G(y,y)g(y,u(y))dy + (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\sum_{k=1}^{m} G(x_k,x_k)I_k(u(x_k))$$
$$\ge (\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\|u\|,\ x \in [0,1].$$

3. MAIN RESULTS

Lemma 3.1: Let $E=(E,\|\cdot\|)$ be a Banach space and let $K \subset E$ be a cone in E and $\|\cdot\|$ be increasing with respect to K .Also, r , R are constants with $0<r<R$. Suppose that $\Phi:(\overline{\Omega}_R\setminus\Omega_r)\bigcap K \to K$ ($\Omega_R=\{u\in E, \|u\|<R\}$) is a continuous, compact map and assume that the conditions are satisfied:

(i) $\|\Phi u\| > x,$ for $u \in \partial\Omega_r \bigcap K$

(ii) $u \ne \mu\Phi(u),$ for $\mu \in [0,1)$ and $u \in \partial\Omega_R \bigcap K$

Then Φ has a fixed point in $K\bigcap\{u\in E: r\le\|u\|\le R\}$.

Proof. In applications below, we take $E=C(I,R)$ and define

$$K=\{u\in C(I,R): u(x)\ge\sigma\|u\|, x\in[a,b]\}.$$

One may readily verify that K is a cone in E . Now,let $r>0$ such that

$$r < min\{\varepsilon_0, \int_0^1 G(\frac{1}{2},y)g(y,\varepsilon_0)dy + \sum_{k=1}^{m} G(\frac{1}{2},x_k)I_k(\varepsilon_0)\} \tag{3.1}$$

and let $R>r$ be chosen large enough later.

Let us define an operator $\Phi:(\overline{\Omega}_R\setminus\Omega_r)\bigcap K\to K$ by

$$(\Phi u)(x)=\int_0^1 G(x,y)g(y,u(y))dy+\sum_{k=1}^{m} G(x,x_k)I_k(u(x_k)),\ x\in I.$$

First we show that Φ is well defined. To see this, notice that if $u \in (\overline{\Omega}_R \setminus \Omega_r) \bigcap K$ then $r \le \|u\| \le R$ and $u(x) \ge (\frac{m(x)}{m(1)} \frac{n(x)}{n(0)})\|u\| \ge (\frac{m(x)}{m(1)} \frac{n(x)}{n(0)})r, 0 \le x \le 1.$ Also notice by (H_1) that

$$g(x,u(x)) \le g(x,(\frac{m(x)}{m(1)} \frac{n(x)}{n(0)})r), \qquad when\ 0 \le u(x) \le r,$$

and

$$g(x,u(x)) \le \max_{r \le u \le R} \max_{0 \le x \le 1} g(x,u) \qquad when\ r \le u(x) \le R.$$

These inequalities with (H_2) guarantee that $\Phi : (\overline{\Omega}_R \setminus \Omega_r) \bigcap K \to K$ is well defined.
Next we show that $\Phi : (\overline{\Omega}_R \setminus \Omega_r) \bigcap K \to K$ .If $u \in (\overline{\Omega}_R \setminus \Omega_r) \bigcap K$, then we have

$$\|\Phi u\| \le \int_0^1 G(y,y)g(y,u(y))dy + \sum_{k=1}^m G(x_k,x_k)I_k(u(x_k))$$

$$\begin{aligned}(\Phi u)(x) &\ge (\frac{m(x)}{m(1)} \frac{n(x)}{n(0)})\int_0^1 G(y,y)g(y,u(y))dy + (\frac{m(x)}{m(1)} \frac{n(x)}{n(0)})\sum_{k=1}^m G(x_k,x_k)I_k(u(x_k)) \\ &\ge (\frac{m(x)}{m(1)} \frac{n(x)}{n(0)})\|\Phi u\|, x \in [0,1].\end{aligned}$$

i.e. $\Phi u \in K$ so $\Phi : (\overline{\Omega}_R \setminus \Omega_r) \bigcap K \to K.$

It is clear that Φ is continuous and completely continuous.
We now show that

$$\|\Phi u\| > \|u\|, \qquad for\ u \in \partial\Omega_r \bigcap K \tag{3.2}$$

To see that,let $u \in \partial\Omega_r \bigcap K$,then $\|u\| = r$ and $u(x) \ge (\frac{m(x)}{m(1)} \frac{n(x)}{n(0)})r$ for $x \in [0,1]$. So by (H_1) and (3.1) we have

$$\begin{aligned}(\Phi u)(\frac{1}{2}) &= \int_0^1 G(\frac{1}{2},y)g(y,u(y))dy + \sum_{k=1}^m G(\frac{1}{2},x_k)I_k(u(x_k)) \\ &\ge \int_0^1 G(\frac{1}{2},y)g(y,r)dy + \sum_{k=1}^m G(\frac{1}{2},x_k)I_k(r)\end{aligned}$$

$$\geq \int_0^1 G(\frac{1}{2}, y) g(y, \varepsilon_0) dy + \sum_{k=1}^{m} G(\frac{1}{2}, x_k) I_k(\varepsilon_0)$$
$$> r = \|u\|.$$

so (3.2) is satisfied.

On the other hand, from (H_4), there exist $0 < \varepsilon < \lambda_1 - f^\infty$ and $H > p$ such that

$$(\lambda_1 - \varepsilon - g^\infty) \int_0^1 (\frac{m(x)}{m(1)} \frac{n(x)}{n(0)}) \varphi_1(x) dx > \sum_{k=1}^{m} (I^\infty(k) + \varepsilon) \varphi_1(x_k);$$

$$g(x,u) \leq (g^\infty + \varepsilon) u, I_k(u) \leq (I^\infty(k) + \varepsilon) u \quad \forall\, x \in [0,1], u \geq H. \tag{3.3}$$

Let $C = \max\limits_{r \leq u \leq H} \max\limits_{0 \leq x \leq 1} g(x,u) + \sum\limits_{k=1}^{m} \max\limits_{r \leq u \leq H} I_k(u),$ it is clear that

$$g(x,u) \leq g(x, (\frac{m(x)}{m(1)} \frac{n(x)}{n(0)}) r) + C + (g^\infty + \varepsilon) u,$$

$$I_k(u) \leq I_k((\frac{m(x)}{m(1)} \frac{n(x)}{n(0)}) r) + C + (I^\infty(k) + \varepsilon) u, \forall\, x \in [0,1], u \geq 0.$$

Next we show that if R is large enough ,then $\mu \Phi u \neq u$ for any $u \in K \bigcap \partial \Omega_R$ and $0 \leq \mu < 1.$ If this is not true ,then there exist $u_0 \in K \bigcap \partial \Omega_R$ and $0 \leq \mu_0 < 1$ such that $\mu_0 \Phi u_0 = u_0$. Thus $\| u_0 \| = R > r$ and $u_0(x) \geq (\frac{m(x)}{m(1)} \frac{n(x)}{n(0)}) R.$ Note that $u_0(x)$ satisfies

$$\begin{cases} Lu_0(x) + \mu_0 g(x, u_0(x)) = 0, \quad x \in I', \\ -\Delta(p u_0') |_{x = x_k} = \mu_0 I_k(u_0(x_k)), \quad k = 1, 2, \cdots, m, \\ \alpha_1 u_0(0) + \beta_1 u'_0(0) = 0 \\ \alpha_2 u_0(1) + \beta_2 u'_0(1) = 0. \end{cases} \tag{3.4}$$

Multiply equation (3.4) by $\varphi_1(x)$ and integrate from 0 to 1, using integration by parts in the left side, notice that

$$\int_0^1 \varphi_1(x)[(p(x)u'_0(x))' + q(x)u_0(x)]dx = \int_0^{x_1} \varphi_1(x)[(p(x)u'_0(x))' + q(x)u_0(x)]dx$$

$$+\sum_{k=1}^{m-1}\int_{x_k}^{x_{k+1}} \varphi_1(x)[(p(x)u'_0(x))' + q(x)u_0(x)]dx + \int_{x_m}^{1} \varphi_1(x)[(p(x)u'_0(x))' + q(x)u_0(x)]dx$$

$$= \varphi_1(x_1)p(x_1)u_0'(x_1-0) - \varphi_1(0)p(0)u_0'(0) - \int_0^{x_1} p(x)u_0'(x)\varphi'_1(x)dx$$

$$+\int_0^{x_1} q(x)u_0(x)\varphi_1(x)dx + \sum_{k=1}^{m-1}[\varphi_1(x_{k+1})p(x_{k+1})u_0'(x_{k+1}-0) - \varphi_1(x_k)p(x_k)u_0'(x_k+0)$$

$$-\int_{x_k}^{x_{k+1}} p(x)u_0'(x)\varphi_1'(x)dx + \int_{x_k}^{x_{k+1}} q(x)u_0(x)\varphi_1(x)dx] + \varphi_1(1)p(1)u'_0(1)$$

$$-\varphi_1(x_m)p(x_m)u_0'(x_m+0) - \int_{x_m}^{1} p(x)u_0'(x)\varphi_1'(x)dx + \int_{x_m}^{1} q(x)u_0(x)\varphi_1(x)dx$$

$$= -\sum_{k=1}^{m}\Delta(p(x_k)u_0'(x_k))\varphi_1(x_k) - \int_0^1 p(x)\varphi_1'(x)u_0'(x)dx + \int_0^1 q(x)\varphi_1(x)u_0(x)dx$$

$$+\varphi_1(1)p(1)u'_0(1) - \varphi_1(0)p(0)u_0'(0).$$

Also notice that

$$\int_0^1 p(x)\varphi_1'(x)u_0'(x)dx = \int_0^1 p(x)\varphi_1'(x)du_0(x)$$

$$= p(1)\varphi_1'(1)u_0(1) - p(0)\varphi_1'(0)u_0(0) - \int_0^1 u_0(x)(p(x)\varphi_1'(x))'dx$$

$$= p(1)\varphi_1'(1)u_0(1) - p(0)\varphi_1'(0)u_0(0) + \int_0^1 u_0(x)q(x)\varphi_1(x)dx$$

$$+\lambda_1\int_0^1 u_0(x)\varphi_1(x)dx.$$

thus, by the boundary conditions, we have

$$\int_0^1 \varphi_1(x)[(p(x)u'_0(x))' + q(x)u_0(x)]dx = -\sum_{k=1}^{m}\Delta(p(x_k)u_0'(x_k))\varphi_1(x_k)$$

$$-p(1)\varphi_1'(1)u_0(1) + p(0)\varphi_1'(0)u_0(0) - \int_0^1 u_0(x)q(x)\varphi_1(x)dx$$

$$-\lambda_1\int_0^1 u_0(x)\varphi_1(x)dx + \int_0^1 q(x)\varphi_1(x)u_0(x)dx + \varphi_1(1)p(1)u'_0(1) - \varphi_1(0)p(0)u_0'(0)$$

$$= -\sum_{k=1}^{m}\Delta(p(x_k)u_0'(x_k))\varphi_1(x_k) - \lambda_1\int_0^1 u_0(x)\varphi_1(x)dx$$

$$= \sum_{k=1}^{m}\mu_0 I_k(u_0(x_k))\varphi_1(x_k) - \lambda_1\int_0^1 u_0(x)\varphi_1(x)dx.$$

So we obtain

$$\begin{aligned}
&\lambda_1\int_0^1 u_0(x)\varphi_1(x)dx=\mu_0\sum_{k=1}^m I_k(u_0(x_k))\varphi_1(x_k)+\mu_0\int_0^1 g(x,u_0(x))\varphi_1(x)dx\\
&\le\sum_{k=1}^m(I^\infty(k)+\varepsilon)\varphi_1(x_k)u_0(x_k)+C\sum_{k=1}^m\varphi_1(x_k)+\sum_{k=1}^m I_k((\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})r)\varphi_1(x_k)\\
&+(g^\infty+\varepsilon)\int_0^1\varphi_1(x)u_0(x)dx+C\int_0^1\varphi_1(x)dx+\int_0^1\varphi_1(x)g(x,(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})r)dx
\end{aligned}$$

Consequently, we obtain that

$$\begin{aligned}
&(\lambda_1-g^\infty-\varepsilon)\int_0^1 u_0(x)\varphi_1(x)dx\le\sum_{k=1}^m(I^\infty(k)+\varepsilon)\varphi_1(x_k)u_0(x_k)\\
&+\int_0^1\varphi_1(x)g(x,(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})r)dx+C(\sum_{k=1}^m\varphi_1(x_k)+\int_0^1\varphi_1(x)dx)+\sum_{k=1}^m I_k((\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})r)\varphi_1(x_k)\\
&\le\|u_0\|\sum_{k=1}^m(I^\infty(k)+\varepsilon)\varphi_1(x_k)+\int_0^1\varphi_1(x)g(x,(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})r)dx\\
&\qquad+C(\sum_{k=1}^m\varphi_1(x_k)+\int_0^1\varphi_1(x)dx)+\sum_{k=1}^m I_k((\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})r)\varphi_1(x_k)
\end{aligned}$$

We also have

$$\int_0^1 u_0(x)\varphi_1(x)dx\ge\|u_0\|\int_0^1(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\varphi_1(x)dx$$

Thus

$$\|u_0\|\le\frac{\int_0^1\varphi_1(x)g(x,\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})r)dx+C(\sum_{k=1}^m\varphi_1(x_k)+\int_0^1\varphi_1(x)dx)+\sum_{k=1}^m I_k((\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})r)\varphi_1(x_k)}{(\lambda_1-g^\infty-\varepsilon)\int_0^1(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})\varphi_1(x)dx-\sum_{k=1}^m(I^\infty(k)+\varepsilon)\varphi_1(x_k)}=:\bar{R}.$$

Let $R>max\{\bar{R},H\}$, then for any $u\in K\bigcap\partial\Omega_R$ and $0\le\mu<1$, we have $\mu\Phi u\ne u$. Hence all the assumptions of Lemma 3.1 are satisfied, Φ has a fixed point u in $K\bigcap\{u\in E:r\le\|u\|\le R\}$, u(x) $\ge(\frac{m(x)}{m(1)}\frac{n(x)}{n(0)})r$, $\forall x\in[0,1]$. Let $\theta^*:=r$, this completes the proof of Theorem 1.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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