Journal of Scientific Research & Reports
3(13): 1815-1827, 2014; Article no. JSRR.2014.13.010

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Bifurcation on Smooth and Non-smooth Traveling Wave Solutions for a Generalized Two-component Camassa-Holm Equation

Xinghua Fan[1*] **and Shasha Li**[1]

[1] *Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, China.*

Original Research Article

Received: 09 April 2014
Accepted: 27 May 2014
Published: 06 June 2014

Abstract

A generalized two-component Camassa-Holm equation is introduced as a model for shallow water waves moving over a linear shear flow. Bifurcations of traveling wave solutions are studied. Phase portraits of the traveling wave system are given. By using the method of planar dynamical systems, the existence of solitary wave solutions, smooth and non-smooth periodic traveling wave solutions is presented in different parametric conditions. Numerical simulations are made to agree the theoretical analysis. It shows that the existence of singular straight lines for the generalized two-component Camassa-Holm equation is the original cause of the non-smooth solutions. The existence of uncountably infinitely many breaking traveling wave solutions are given.

Keywords: Generalized two-component Camassa-Holm equation; bifurcation; traveling wave solution; smoothness of wave; peakon;

2010 Mathematics Subject Classification: 35Q51; 35Q58;37K50

1 Introduction

In this study, we consider the following generalized two-component Camassa-Holm equation

$$u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_x u_{xx} + uu_{xxx}) + \epsilon\rho\rho_x = 0, \tag{1.1a}$$

$$\rho_t + (\rho u)_x = 0. \tag{1.1b}$$

where the variable $u(x, t)$ represents the horizontal velocity of the fluid and the variable $\rho(x, t)$ is related to the free surface elevation from equilibrium, the constant $A \geq 0$ characterizes a linear underlying shear flow, σ is a real parameter and $\epsilon = \pm 1$. When $\epsilon = 1$, system (1.1) was derived by Chen and Liu from shallow water theory with nonzero constant vorticity [1]. The case $\epsilon = -1$ corresponds to the situation in which the gravity acceleration points upwards.

*Corresponding author: E-mail: fan131@ujs.edu.cn

The generalization in Eq. (1.1) can be seen from the point of view of both a single equation and a two-component system. Firstly, Eq. (1.1) is a two-component generalization of the interesting Camassa-Holm (CH) equation [2]

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.2}$$

When $\rho \equiv 0$ and $A = 0$, the CH equation (1.2) can be recovered from (1.1). The CH equation is used to model many nonlinear phenomena such as the propagation of unidirectional irrotational shallow water waves over a flat bed [2] and water waves moving over an underlying shear flow [5].An important feature of the CH equation is the presence of the "peakons" [2]: $u(x,t) = ce^{|x-ct|}, c \neq 0$, which are smooth except at the crests, where they are continuous, but have a jump discontinuity in the first derivative.

When $\rho \equiv 0$, system (1.1) turns to be

$$u_t - u_{xxt} + 3uu_x = \sigma(2u_x u_x x + uu_{xxx}), \tag{1.3}$$

which models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods ([3],[4]). The authors showed the appearance of supersonic solitary shock waves because there is a vertical singular line in the phase plane [4]. Those waves have discontinuous first-order derivative at the peak and we point out that they are actually peakons. We will point out that this feature is kept in system (1.1) since there are two vertical singular lines.

Secondly, Eq. (1.1) is a generation of some two-component Camassa-Holm equations. There are many integrable two-component Camassa-Holm equations ([6],[7],[8],[9],[10],[11], [12],[13],[14],[15]). The most popular one is

$$\begin{cases} m_t + 2mu_x + um_x + \epsilon\rho\rho_x = 0, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{1.4}$$

where $m = u - u_{xx}$. System (1.4) is integrable[9] as it can be written as a compatibility condition of two linear systems (Lax pair) with a spectral parameter.

For $\epsilon = -1$, the integrability of (1.4) was proved in [16]. Global existence and blow-up phenomena of (1.4) were studied in [6]. Bifurcation of traveling wave solutions of (1.4) was given in [10]. The geometry of (1.4) was studied in [17]. The local well-posedness for a periodic two-component Camassa-Holm equation was established in [18]. Wave breaking phenomenon is of special interest ([14],[15]).

Compared with (1.4), there is a new free parameter σ in (1.1). Clearly, when $\sigma = 1$, (1.1) turns to be (1.4).

It was shown in [10] that when $\sigma = 1$, the two component system (1.1) has many interesting traveling wave solutions such as smooth solitary waves, kink and anti-kink wave solutions. But no peakon solutions were found. We point out that peakon solutions exists when $\sigma > 1$.

In this work, we are interested in finding the dynamical behavior of the traveling wave solutions of (1.1). We will show that the new parameter σ has some matters to the type of traveling wave solutions. Noticing that the term Au_x can be canceled by a Galileo transformation, we assume $A = 0$ in the rest of the text.

Let

$$u(x,t) = \phi(\xi), \rho(x,t) = \psi(\xi), \quad \xi = x - ct, \tag{1.5}$$

where c is the wave speed. Substituting (1.5) into (1.1b), we have

$$-c\psi' + (\phi\psi)' = 0, \tag{1.6}$$

where "$'$" is the derivative with respect to ξ. Integrating (1.6) once and setting the integration constant as r, $r \neq 0$, we get

$$\psi = \frac{r}{\phi - c}. \tag{1.7}$$

Substituting (1.5) into (1.1a) we have

$$-c\phi' + c\phi''' - \sigma\,\phi\phi''' - 2\,\sigma\,\phi'\phi'' + 3\,\phi\phi' + \epsilon\psi\psi' = 0. \tag{1.8}$$

Noticing (1.7), integrating (1.8) once with respect to ξ and setting the integrating constant as zero, we have

$$\frac{3}{2}\phi^2 - c\phi - \frac{1}{2}\sigma(\phi')^2 + (-\sigma\,\phi + c)\,\phi'' + \frac{\epsilon r}{2(\phi-c)^2} = 0. \tag{1.9}$$

Eq. (1.9) is equivalent to the planar system

$$\begin{aligned} \frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= \frac{-\sigma(\phi-c)^2y^2 + \phi(\phi-c)^2(3\phi-2c) + \epsilon r^2}{2(\phi-c)^2(\sigma\phi-c)}. \end{aligned} \tag{1.10}$$

Since a phase orbit defined by the vector fields of system (1.10) determines a traveling wave solution of (1.1), we shall investigate the bifurcations of the phase portraits of (1.10) in the phase plane as the parameters are changed. Without loss of generality, we assume $c > 0$ and $\sigma > 0$. Here we only consider the bounded solutions. We shall apply the bifurcation theory of dynamical systems ([19],[20]) in this study.

The rest of the paper is organized as follows. Section 2 gives bifurcations conditions of (2.1) and different phase portraits associated with different parameters. Section 3 concerns the existence of smooth and non-smooth traveling wave solutions of (1.1). Section 4 devotes to the conclusions.

2 Bifurcation conditions and possible phase portraits

In this section, we will first transfer the singular system (1.10) to its equivalent non-singular system. Then we will determine the number and type of possible equilibrium points of the non-singular system. Finally, we will present possible phase portraits of the non-singular system.

Clearly, system (1.10) has two singular line $\phi = \phi_s = c$ and $\phi = \phi_\sigma = \frac{c}{\sigma}$. On those two straight lines of the phase plane (ϕ, y), ϕ'' is not well defined. To avoid the singularity, let $d\xi = 2(\phi-c)^2(\sigma\phi-c)d\tau$ for $\phi \neq c$ and $\phi \neq \phi_\sigma$. Then system (1.10) becomes

$$\begin{aligned} \frac{d\phi}{d\tau} &= 2(\phi-c)^2(\sigma\phi-c)y, \\ \frac{dy}{d\tau} &= -\sigma(\phi-c)^2y^2 + \phi(\phi-c)^2(3\phi-2c) + \epsilon r^2. \end{aligned} \tag{2.1}$$

For system (2.1), the singular straight lines $\phi = c$ and $\phi = \phi_\sigma$ become two invariant straight line solutions. In the sense of the theory of geometric singular perturbation, the variable τ is a fast variable near the straight lines, while the variable ξ is a slow variable.

System (2.1) has the first integral as system (1.10)

$$H = (\sigma\phi - c)y^2 - \frac{\phi^2(\phi-c)^2 - \epsilon r^2}{\phi - c}. \tag{2.2}$$

We now investigate the bifurcations of phase portraits of system (2.1). Let

$$f(\phi) = \phi\,(\phi-c)^2\,(3\,\phi - 2\,c) + \epsilon r^2.$$

Then $f'(\phi) = -2\,(c-\phi)\left(6\,\phi^2 - 6\,c\phi + c^2\right)$ has three roots $\phi^* = c, \phi_\pm = (\frac{1}{2} \pm \frac{\sqrt{3}}{6})c$.It follows that $f(c) = \epsilon r^2$, $f(\phi_+) = \left(-\frac{1}{12} + \frac{\sqrt{3}}{18}\right)c^4 + \epsilon r^2$,$f(\phi_-) = \left(-\frac{1}{12} - \frac{\sqrt{3}}{18}\right)c^4 + \epsilon r^2$,$f(\phi_\sigma) = \frac{c^4(\sigma-1)^2(3-2\,\sigma)+\epsilon r^2\sigma^4}{\sigma^4}$.

Equilibrium points of (2.1) include regular and singular points. Clearly, on the $\phi-$axis, system (2.1) has at most 4 equilibrium points $E_i(\phi_i, 0), i = 1, 2, 3, 4$. On the straight line $\phi = c$, there is no equilibrium point. On the straight line $\phi = \frac{c}{\sigma}$ there are at most two singular equilibrium points. If $Y_s > 0$, then there exist two singular equilibrium points $Q_\pm(\phi_\sigma, \pm\sqrt{Y_s})$ where $Y_s = \frac{c^4(3-2\sigma)(\sigma-1)^2+\epsilon r^2\sigma^4}{c^2\sigma^3(\sigma-1)^2}$. No other equilibrium points are found.

Let $M(\phi_i, 0)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point $E_i(\phi_i, 0)$. Then we have

$$J(\phi_i, 0) = detM(\phi_i, 0) = -2\sigma\ (c-\phi)^2\ (\phi_i - \phi_\sigma\ )\ f'(\phi_i). \tag{2.3}$$

By the theory of planar dynamical systems [20], for an equilibrium point of a planar integral system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$, then it is a center; if $J = 0$ and the Poincar$\acute{e}$ index of the equilibrium point is zero, then it is a cusp.

From (2.3) we see that the types of the equilibrium points $E_i(\phi_i, 0)$ of system (2.1) are determined by the sign of $f'(\phi_i)$ and the relative positions of the equilibrium points with respect to the singular straight line $\phi = \phi_\sigma$.

Letting $\phi_- = \phi_\sigma$ we have the parameter condition $\sigma = \sigma^* = 3 + \sqrt{3}$. Letting $\phi_+ = \phi_\sigma$ we get another condition $\sigma = \sigma_0 = 3 - \sqrt{3}$.

2.1 Phase portraits of system (2.1) when $f(\phi_-) < 0, f(\phi_+) > 0$, $\epsilon = -1$

Proposition 1. When $f(\phi_-) < 0, f(\phi_+) > 0$, $\epsilon = -1$, there are four regular equilibrium points for system (2.1).

1. If $0 < \sigma < 1$ and $f(\phi_\sigma) < 0$, then $\phi_1 < \phi_- < \phi_2 < \phi_+ < \phi_3 < c < \phi_\sigma < \phi_4$. E_1, E_3 and E_4 are saddle points while E_2 is a center. There is a homoclinic orbit to E_3 enclosing the center E_2. There is a family of closed orbits surrounding the center E_2.

2. If $0 < \sigma < 1$ and $f(\phi_\sigma) > 0$, then $\phi_1 < \phi_- < \phi_2 < \phi_+ < \phi_3 < c < \phi_4 < \phi_\sigma$. E_2 and E_4 are centers while E_1 and E_3 are saddle points. There are two equilibrium points $Q_\pm$ on the singular line $\phi = \phi_\sigma$. There is a homoclinic orbit to E_3 enclosing the center E_2. There are two families of closed orbits surrounding the center E_2 and E_4, respectively.

3. If $1 < \sigma < \sigma_0$ and $f(\phi_\sigma) < 0$, then $\phi_1 < \phi_- < \phi_2 < \phi_+ < \phi_3 < \phi_\sigma < c < \phi_4$. E_2 is a centers E_1, E_3 and E_4 are saddle points. There is a homoclinic orbit to E_3 enclosing the center E_2. There is a family of closed orbits surrounding the center E_2.

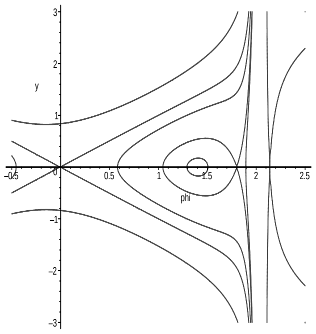
4. If $1 < \sigma < \sigma_0$ and $f(\phi_\sigma) > 0$, then $\phi_1 < \phi_- < \phi_2 < \phi_+ < \phi_\sigma < \phi_3 < c < \phi_4$. There are two equilibrium points on the singular line $\phi = \phi_\sigma$. E_2 and E_3 are centers, E_1 and E_4 are saddle points.

5. If $\sigma_0 < \sigma < \sigma^*$ and $f(\sigma) < 0$ then $\phi_1 < \phi_- < \phi_\sigma < \phi_2 < \phi_+ < \phi_3 < c < \phi_4$. E_1 , E_2 and E_4 are saddle points while E_3 is a center. There is a homoclinic orbit to E_2 enclosing the center E_3. There is a family of closed orbits surrounding the center E_3.

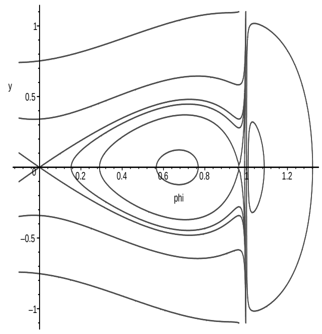
6. If $\sigma_0 < \sigma < \sigma^*$ and $f(\sigma) > 0$ then $\phi_1 < \phi_- < \phi_2 < \phi_\sigma < \phi_+ < \phi_3 < c < \phi_4$. There are two equilibrium points on the singular line $\phi = \phi_\sigma$. E_1 and E_4 are saddle points. E_2 and E_3 are centers.

7. If $\sigma > \sigma^*$ then $\phi_1 < \phi_\sigma < \phi_- < \phi_2 < \phi_+ < \phi_3 < c < \phi_4$. E_1, E_2 and E_4 are saddle points while E_3 is a center. There is a homoclinic orbit to E_2 enclosing the center E_3. There is a family of closed orbits surrounding the center E_3.

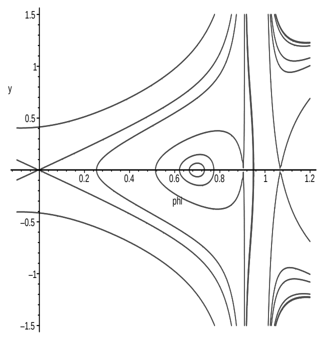
In this case, phase portraits of system (2.1) are shown in Fig.1.



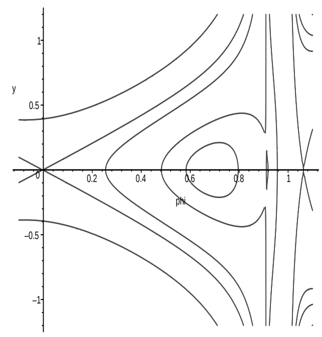
(a) $0 < \sigma < 1,\, f(\phi_\sigma) < 0$



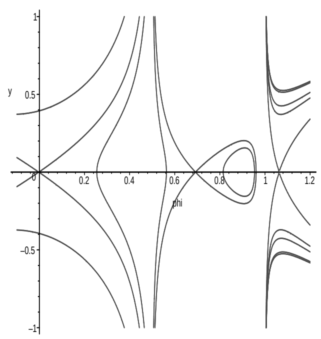
(b) $0 < \sigma < 1,\, f(\phi_\sigma) > 0,$



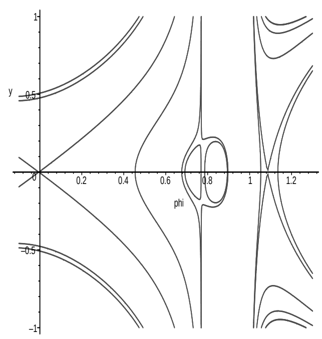
(c) $1 < \sigma < \sigma_0, f(\phi_\sigma) < 0$



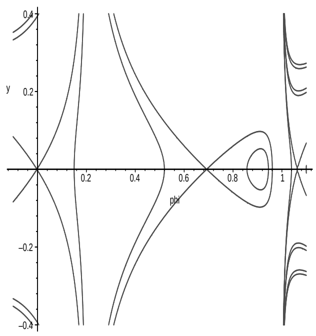
(d) $1 < \sigma < \sigma_0, f(\phi_\sigma) > 0$



(e) $\sigma_0 < \sigma < \sigma^*, f(\phi_\sigma) < 0$



(f) $\sigma_0 < \sigma < \sigma^*, f(\phi_\sigma) > 0$



(g) $\sigma > \sigma^*$

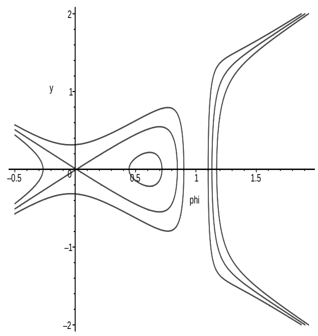
Figure 1: Phase portraits of (2.1) when $f(\phi_-) < 0, f(\phi_+) > 0$, $\epsilon = -1$

2.2 Phase portraits of system (2.1) when $f(\phi_-) < 0, f(\phi_+) > 0$,$\epsilon = 1$

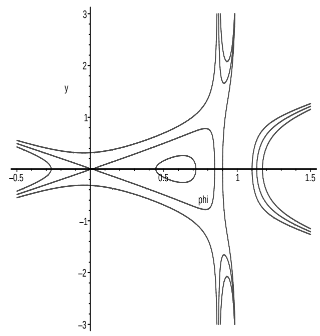
Proposition 2. When $f(\phi_-) < 0, f(\phi_+) > 0$,$\epsilon = 1$, there are two regular equilibrium points for system (2.1). No other equilibrium points exist.

1. If $0 < \sigma < 1$, then $\phi_1 < \phi_- < \phi_2 < \phi_+ < c < \phi_\sigma$. E_1 is a saddle point and E_2 is a center. There is a homoclinic orbit to E_1 enclosing the center E_2.
2. If $1 < \sigma < \sigma_0$, then $\phi_1 < \phi_- < \phi_2 < \phi_+ < \phi_\sigma < c$. E_1 is a saddle point and E_2 is a center. There is a homoclinic orbit to E_1 enclosing the center E_2.
3. If $\sigma_0 < \sigma < \sigma^*$ and $f(\phi_\sigma) > 0$, then $\phi_1 < \phi_- < \phi_2 < \phi_\sigma < \phi_+ < c$. E_1 is a saddle point while E_2 is a center.
4. If $\sigma_0 < \sigma < \sigma^*$ and $f(\phi_\sigma) < 0$, then $\phi_1 < \phi_- < \phi_\sigma < \phi_2 < \phi_+ < c$. Both E_1 and E_2 are saddle points.
5. If $\sigma > \sigma^*$ and $f(\phi_\sigma) < 0$, then $\phi_1 < \phi_\sigma < \phi_- < \phi_2 < \phi_+ < c$. Both E_1 and E_2 are saddle points.
6. If $\sigma > \sigma^*$ and $f(\phi_\sigma) > 0$, then $\phi_\sigma < \phi_1 < \phi_- < \phi_2 < \phi_+ < c$. E_1 is a center and E_2 is a saddle point.

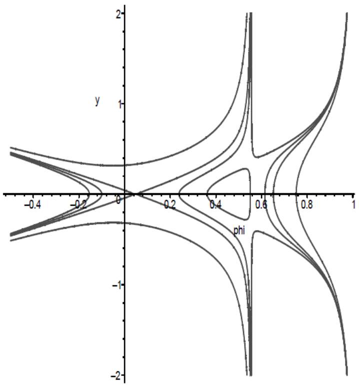
In this case, phase portraits of system (2.1) are shown in Fig. 2.



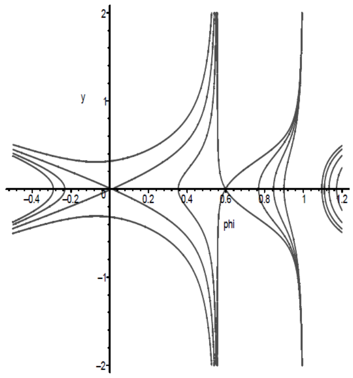
(a) $0 < \sigma < 1$



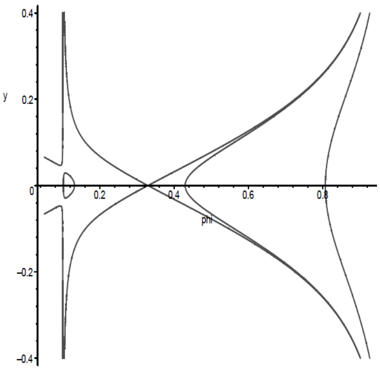
(b) $1 < \sigma < \sigma_0$



(c) $\sigma_0 < \sigma < \sigma^*, f(\sigma) > 0$



(d) $\sigma_0 < \sigma < \sigma^*, f(\sigma) < 0$ or $\sigma > \sigma^*$, $f(\phi_\sigma) < 0$



(e) $\sigma > \sigma^*, f(\sigma) > 0$

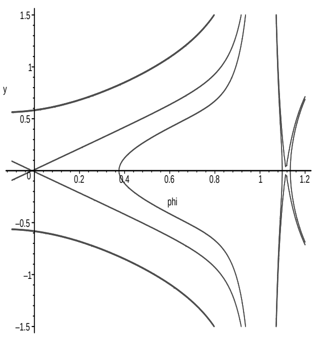
Figure 2: Phase portraits of system (2.1) when $f(\phi_-) < 0, f(\phi_+) > 0$,$\epsilon = 1$

2.3 Phase portraits of system (2.1) when $f(\phi_-) < 0, f(\phi_+) < 0$, $\epsilon = -1$

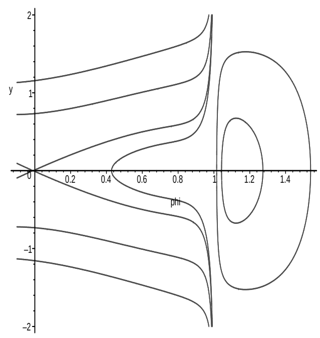
Proposition 3. When $f(\phi_-) < 0, f(\phi_+) < 0$, $\epsilon = -1$, there are two regular equilibrium points for system (2.1). No other equilibrium points exist.

1. If $0 < \sigma < 1$ and $f(\sigma) < 0$, then $\phi_1 < \phi_- < \phi_+ < c < \phi_\sigma < \phi_2$. Both E_2 and E_1 are saddle points.
2. If $0 < \sigma < 1$ and $f(\sigma) > 0$, then $\phi_1 < \phi_- < \phi_+ < c < \phi_2 < \phi_\sigma$. E_2 is a center, E_1 is a saddle point.
3. If $\sigma > 1$ then $\phi_1 < \phi_\sigma < \phi_- < \phi_+ < c < \phi_2$. Both E_2 and E_1 are saddle points.

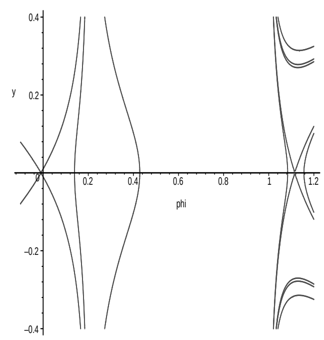
In this case, phase portraits of system (2.1) are shown in Fig. 3.



(a) $0 < \sigma < 1$, $f(\sigma) < 0$,



(b) $0 < \sigma < 1$, $f(\sigma) > 0$,



(c) $\sigma > 1$,

Figure 3: Phase portraits of (2.1) when $f(\phi_-) < 0, f(\phi_+) < 0$, $\epsilon = -1$

3 The existence of smooth and non-smooth traveling wave solutions

We will discuss the existence of smooth and non-smooth traveling wave solutions. We first consider the existence of smooth wave solutions of (1.1).

We denote that $h_i = H(\phi_i, 0)$, $H^* = H(\phi_\sigma, \pm\sqrt{Y_s})$ defined by (2.2).

Usually, a solitary wave solution of (1.1) corresponds to a homoclinic orbit of system (1.10). A periodic orbit of system (1.10) corresponds to a periodically traveling wave solution of (1.1). Thus, we need to find all periodic annuli and homoclinic orbits of system (1.10).

From Propersition 1 (1)-(3), we see the following theorem holds.

Theorem 1. Suppose that $\epsilon = -1$ and $0 < \sigma < 1$ (or $1 < \sigma < \sigma_0, f(\phi_\sigma) < 0$) . Then, corresponding to $H(\phi, y) = h_3$, (1.1) has a smooth solitary traveling wave solution of valley type. Corresponding to the closed curves $H(\phi, y) = h, h \in (h_3, h_2)$ defined by (2.2), Eq. (1.1) has a family of smooth periodic wave solutions (see Fig. 1(a),1(b),1(c)).

The homoclinic orbit $H(\phi, y) = h_3$ to the saddle point E_3 intersects the $\phi-$axis at the point $(\phi_m, 0)$. Let ϕ_a and ϕ_b be zeroes of the polynomial $(\phi - c)H_3 + \phi^2(\phi - c)^2 - \epsilon r^2$ besides ϕ_3. We see from (2.2) that the homoclinic orbit to the saddle point E_3 can be expressed by

$$y = \pm\frac{(\phi_3 - \phi)\sqrt{(\phi - \phi_a)(\phi - \phi_b)}}{\sqrt{(\phi - c)(\sigma\phi - c)}}, \phi_m < \phi < \phi_3, \tag{3.1}$$

By using the first equation of (1.10) and taking initial value $\phi(0) = \phi_0$, on a branch of the homoclinic orbit to do integration, we have the implicit expression of the smooth solitary solution

$$\int_{\phi_0}^{\phi} \frac{\sqrt{(s - c)(\sigma s - c)}}{(\phi_3 - s)\sqrt{(s - \phi_a)(s - \phi_b)}} ds = \pm\xi. \tag{3.2}$$

We carry numerical simulations to draw the portrait of the solution. Taking parameters $c = 2, \sigma = 0.95, r^2 = 0.1, \epsilon = -1$, the one-dimensional portrait of the smooth solitary traveling wave solution is shown in Fig. 4(c)

Next we will consider the non-smooth solutions. Because of the existence of singular lines $\phi = c$ and $\phi = \phi_\sigma$, system (1.1) sometimes has non-smooth traveling wave solutions. This phenomenon has been studied by some authors (see [20, 21]).

Take Fig.1(b) as an example. Denote the closed orbit $H(\phi, y) = h, h \in (H^*, h_4)$ as Γ^h. From Lemma 4.1 in Ref. [21], we see that for $h < h_4$, as h decreases and approaches H^*, a segment of the arcs of the orbits of perotic families surrounding the center $E_4(\phi_4, 0)$ will accumulate into a segment S_1S_2 on the straight line $\phi = \phi_\sigma$. It means that in a very short time interval of ξ, $y = \phi'$ changes its sign rapidly and ϕ rapidly changes its motion direction to form a profile of cusp wave. Then, we have the following conclusion.

Theorem 2. Suppose that $\epsilon = -1$, $0 < \sigma < 1$ and $f(\phi_\sigma) > 0$. Then, corresponding to the closed curves $H(\phi, y) = h, h \in (H^*, h_4)$ defined by (2.2), Eq. (1.1) has a family of periodic traveling wave solutions. When h decreases from h_4 to H^*, these periodic traveling wave will gradually lose their smoothness, and evolve from smooth periodic traveling wave to periodic cusp traveling wave. (see Fig. 1(b)).

Remark 3.1. Similar conclusions can be made for Figs. 1(d), 1(f), 2(c),2(e) and 3(b). For simplicity, we only give the scope of the constant h in the expression of the closed curves $H(\phi, y) = h$. In Fig. 1(d), smooth periodic becomes periodic cusp waves as h increases from h_3 to H^*. In Fig.1(f), there are two families of smooth periodic waves turning to non-smooth periodic cusp waves. One is as h decreases from h_2 to H^*, another is as h increases from h_3 to H^*. In Fig. 2(e), smooth periodic becomes periodic cusp waves as h increase from h_1 to H^*. In Figs.2(c) and 3(b), smooth periodic becomes periodic cusp waves as h decreases from h_2 to H^*.

For Fig. 1(e), the homoclinic orbit has an intersection point $(\phi_M, 0)$ near the singular line. Thus, we have the following theorem:

Theorem 3. Suppose that $f(\phi_-) < 0, f(\phi_+) > 0$, $\epsilon = -1$,$\sigma_0 < \sigma < \sigma^*, f(\phi_\sigma) < 0$. Then, $H(\phi, y) = h_2$ has a zero ϕ_M satisfying $\phi_3 < \phi_M < c$, and there is a smooth solitary wave of bell type of (1.1). For $h \in (h_3, h_2)$, there is a family of periodic traveling wave solutions defined by a branch of $H(\phi, y) = h$. When h increase from h_3 to h_2, these periodic traveling waves will gradually lose their smoothness, and evolve from smooth periodic traveling wave to periodic cusp traveling wave and approach to a peakon (see Fig.1(e)).

Remark 3.2. Similar conclusions can be made for Figs. 1(g), 2(a) and 2(b) because every homoclinic orbit has an intersection point on the $\phi-$axis near one of the singular line $\phi = c$ or $\phi = \phi_\sigma$. We point out that the scope of the constant h are different. In Fig. 2(a), h decrease from h_2 to h_1. In Fig. 1(g), h increase from h_3 to h_2. In Fig. 2(b), h decrease from h_2 to h_1.

As a numerical simulation result, we show the above mentioned smooth and on-smooth traveling wave solutions in Fig. 4. Parameters are taken as $c = 1, \sigma = 0.5, r^2 = 0.001$ for the periodic wave to periodic cusp wave case. For the peakon case, parameters are $c = 1, \sigma = 5, r^2 = 0.05, \epsilon = -1$. We only show the smooth solitary wave with valley form since the bell type can be seen as the upside-down of it.

We will study the bounded breaking wave solutions of (1.1) in the end. A breaking wave is a solution which remains bounded while its slope becomes unbounded in finite time. As Lemma 5 in Ref. [10] shows, if along an open orbit of system (2.1), ϕ approaches to the vertical line $\phi = c$ or $\phi = \phi_\sigma$ in positive and negative directions, respectively, then the wave slope becomes unbounded in finite time.

If an open orbit has an intersection point on the $\phi-$axes and stretches up and down vertically, then the corresponding solution is a break wave. Corresponding to those phase portraits in Fig.1-3, we see when $\epsilon = 1$,if $\sigma > 1$, there exist uncountably infinitely many bounded breaking wave solutions of the generalized Camassa-Holm equation (1.1); if $0 < \sigma < 1$, there are only smooth wave solutions. When $\epsilon = -1$, (1.1) has uncountably infinitely many bounded breaking wave solutions for both $\sigma > 1$ and $0 < \sigma < 1$.

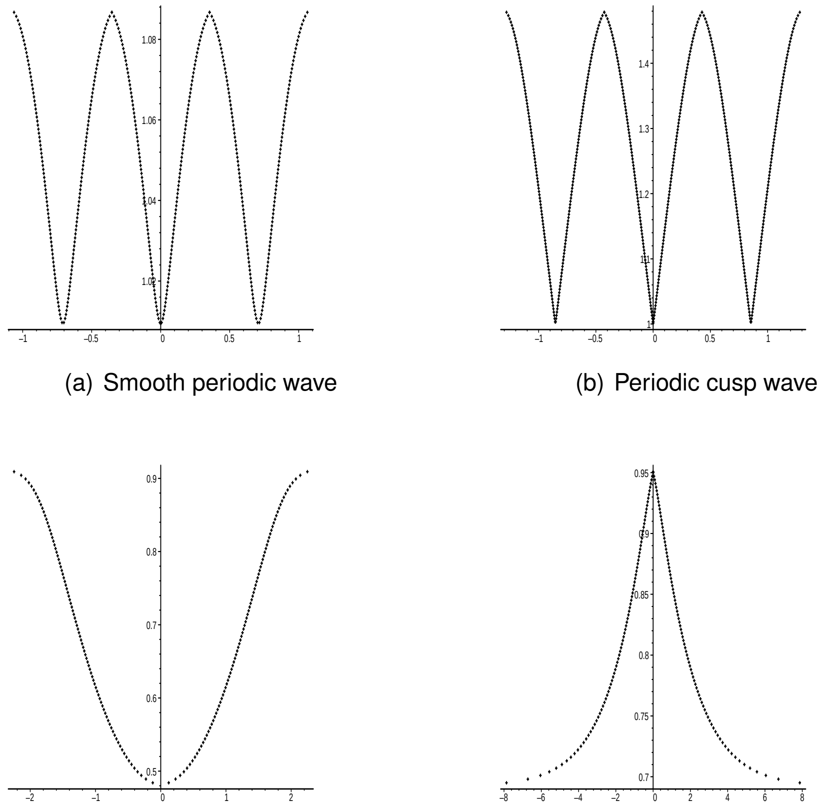
Generally we have the following theorem:

Theorem 4. If system (2.1) has only two regular equilibrium points $E_i(\phi_1, 0), i = 1, 2$ satisfying $\phi_1 < c < \phi_2$ or $\phi_1 < \phi_\sigma < \phi_2$, then, these equilibrium are saddle points and (1.1) has uncountably infinitely many bounded breaking wave solutions.

Remark 3.3. When system (2.1) has four equilibrium points (like Figs. 1(e) and 1(f), Eq. (1.1) also has uncountably infinitely many breaking wave solutions.

4 Conclusions

A generalized two-component Camassa-Holm equation was been presented in this paper. The modification of the sign of $\rho\rho_x$ as well as the free new parameter σ played an important role in the type of traveling wave solutions. Taking plus sign, there were only smooth traveling wave solutions when $\sigma < 1$. In other cases, non smooth traveling wave solutions, such as peakon, periodic cusp solutions appeared. It is interesting to see that just like the remarkable Camassa-Holm equation, the generalized two-component Camassa-Holm equation posses both peakons and the wave breaking phenomena. However the affection of the sign on the integrability of the two-component equation remains further study.



(a) Smooth periodic wave

(b) Periodic cusp wave

(c) Solitary with valley form

(d) Peakon

Figure 4: Different kind of traveling wave solutions of (1.1)

Acknowledgements

Research was supported by the National Natural Science Foundation of China (No. 11026169).

Competing Interests

The authors declare that no competing interests exist.

References

[1] Robin Ming Chen, Yue Liu. Wave breaking and global existence for a generalized two-component Camassa-Holm system, Inter. Math. Res. Not. 2011;6:1381-1461.

[2] Roberto Camassa, Darryl D. Holm, James M. Hyman. An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. 1993;71:1661-1664.

[3] Dai H H. Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod. Acta Mech. 1998;127:193-207.

[4] Dai HH, Huo Y. Solitary shock waves and other travelling waves in a general compressible hyperelastic rod. Proc. R. So.c Lond. 2000;456(1994):331-363.

[5] Robin S. Johnson. The Camassa-Holm equation for water waves moving over a shear flow. *Fluid Dyn Res*, 33(1-2):97 – 111, 2003. In memoriam: Prof. Philip Gerald Drazin 1934-2002.

[6] Chunxia Guan and Zhaoyang Yin. Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system. J. Differ. Equ. 2010;248(8):2003-2014.

[7] Guilong Gui, Yue Liu. On the global existence and wave-breaking criteria for the two-component Camassa-Holm system. J. Funct. Anal. 2010;258(12):4251-4278.

[8] Octavian G. Mustafa. On smooth traveling waves of an integrable two-component Camassa-Holm shallow water system. Wave Motion. 2009;46(6):397-402.

[9] Adrian Constantin, Rossen I. Ivanov. On an integrable two-component Camassa-Holm shallow water system. Phys. Lett. A. 2008;372(48):7129-7132.

[10] Ji Bin Li, Yi Shen Li. Bifurcation of travelling wave solutions for a two-component Camassa-Holm equation. Acta Math. Sin. 2008;24(8):1319-1330.

[11] Shaolong Xie, Yuzhong Zhang, Jianghua He. Two types of bounded traveling-wave solutions of a two-component Camassa-Holm equation. Appl. Math. Comput. 2013;219(20):10271–10282.

[12] Ying Zhang. Global weak solutions for a periodic two-component -Camassa-Holm system. Math. Meth. Appl. Sci. 2013;36(13):1734-1745.

[13] Katrin Grunert, Helge Holden, and Xavier Raynaud. Global dissipative solutions of the two-component Camassa-Holm system for initial data with nonvanishing asymptotics. Nonlinear Anal. Real World Appl., 17:203-244, 2013.

[14] Yanyi Jin, Zaihong Jiang. Wave breaking of an integrable Camassa-Holm system with two components. Nonlinear Anal. Theo. Meth. Appl. 2014;95:107-116.

[15] Zhengguang Guo, Mingxuan Zhu. Wave Breaking and Measure of Momentum Support for an Integrable Camassa-Holm System with Two Components. Stud. Appl. Math. 2013;130(4):417-430.

[16] Ming Chen, Si-Qi Liu, Youjin Zhang. A two-component generalization of the Camassa-Holm equation and its solutions. Lett. Math. Phys. 2006;75(1):1-15.

[17] Escher J, Kohlmann M, Lenells J. The geometry of the two-component Camassa-Holm and Degasperis-Procesi equations. J. Geom. Phys. 2011;61(2):436-452.

[18] Qiaoyi Hu, Zhaoyang Yin. Well-posedness and blow-up phenomena for a periodic two-component Camassa-Holm equation. Proc. Roy. Soc. Edinburgh Sect. A. 2011;141(1):93-107.

[19] Perko L. Differential Equations and Dynamical Systems. Springer-Verlag, New York; 1991.

[20] Li JB, Dai HH. On the Study of Singular Nonlinear Travelling Wave Equations: Dynamical Approach. Science Press, Beijing; 2007.

[21] Jibin Li, Zhengrong Liu. Smooth and non-smooth traveling waves in a nonlinearly dispersive equation. Appl. Math. Model. 2000;25(1):41-56.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=527&id=22&aid=4839