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Concerning the Navier-Stokes problem

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Abstract: The problem discussed is the Navier-Stokes problem (NSP) in $\mathbb{R}^3$. Uniqueness of its solution is proved in a suitable space X. No smallness assumptions are used in the proof. Existence of the solution in X is proved for $t \in [0, T]$, where $T > 0$ is sufficiently small. Existence of the solution in X is proved for $t \in [0, \infty)$ if some a priori estimate of the solution holds.

Keywords: Navier-Stokes equations, uniqueness of the solutions.

MSC: 35Q30, 76D05.

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1. Introduction

There is a large literature on the Navier-Stokes problem (NSP) in $\mathbb{R}^3$ (see [1], Chapter 5) and references therein). The global existence and uniqueness of a solution in $\mathbb{R}^3$ was not proved. The goal of this paper is to prove uniqueness of the solution to NSP in a suitable functional space. No smallness assumptions are used in our proof.

The NS problem in $\mathbb{R}^3$ consists of solving the equations

$$v' + (v, \nabla)v = -\nabla p + \nu \Delta v + f, \quad x \in \mathbb{R}^3,\ t \geq 0, \quad \nabla \cdot v = 0, \quad v(x, 0) = v_0(x). \tag{1}$$

Vector-functions $v = v(x,t)$, $f = f(x,t)$ and the scalar function $p = p(x,t)$ decay as $|x| \to \infty$ uniformly with respect to $t \in \mathbb{R}_+ := [0, \infty)$, $v' := v_t$, $\nu = const > 0$ is the viscosity coefficient, the velocity v and the pressure p are unknown, v_0 and f are known, $\nabla \cdot v_0 = 0$. Equations (1) describe viscous incompressible fluid with density $\rho = 1$.

We use the integral equation for v:

$$v(x,t) = F - \int_0^t ds \int_{\mathbb{R}^3} G(x-y, t-s)(v, \nabla)v dy. \tag{2}$$

Equation (2) is equivalent to (1), see [2]. Formula for the tensor G is derived in [2], see also [1], p.41. The term $F = F(x,t)$ depends only on the data f and v_0 (see equation (18) in [2] or formula (5.42) in [1]):

$$F := \int_{\mathbb{R}^3} g(x-y)v_0(y)dy + \int_0^t ds \int_{\mathbb{R}^3} G(x-y, t-s)f(y,s)dy. \tag{3}$$

We assume throughout that f and v_0 are such that F is bounded in all the norms we use.

Let X be the Banach space of continuous functions with respect to t with the norm

$$\|\tilde{v}\| := \int_{\mathbb{R}^3} |\tilde{v}(\xi, t)|(1 + |\xi|)d\xi, \tag{4}$$

where $t > 0$, and $\tilde{v} := (2\pi)^{-3} \int_{\mathbb{R}^3} v(x,t) e^{-i\xi \cdot x} dx$. Taking the Fourier transform of (2) yields

$$\tilde{v} = \tilde{F} - \int_0^t ds \tilde{G} \tilde{v} \star i\xi \tilde{v} := B(\tilde{v}), \tag{5}$$

where $\star$ denotes the convolution in $\mathbb{R}^3$ and for brevity we omitted the tensorial indices: instead of $\tilde{G}_{mp}\tilde{v}_j \star (i\xi_j)\tilde{v}_p$, where one sums up over the repeated indices, we wrote $\tilde{G}(\xi, t-s)\tilde{v} \star (i\xi\tilde{v})$. From formula (5.9) in [1] it follows that

$$|\tilde{G}| \le ce^{-\nu\xi^2(t-s)}. \tag{6}$$

By $c > 0$ we denote various constants independent of t and ξ. Let $S(\mathbb{R}^3 \times \mathbb{R}_+)$ and $S(\mathbb{R}^3)$ be the L.Schwartz spaces. Our results are:

Theorem 1. *Assume that f and v_0 are in $S(\mathbb{R}^3 \times \mathbb{R}_+)$ and $S(\mathbb{R}^3)$ respectively. Then there is at most one solution to NSP in X.*

Theorem 2. *The solution to NSP in X exists for $t \in [0, T]$ if $T > 0$ is sufficiently small.*

Theorem 3. *The solution $v(x,t)$ to NSP in X exists for all $t \ge 0$ if an a priori estimate $\sup_{t\ge 0} \|\tilde{v}(\xi,t)\| < c_a$ holds, where $c_a > 0$ is a constant depending only on the data.*

2. Proofs

Proof of Theorem 1. Let $\tilde{v}$ and $\tilde{w}$ belong to X and solve equation (5). Denote $z := \tilde{v} - \tilde{w}$. Then (5) implies

$$z = -\int_0^t ds\tilde{G}(z \star i\xi\tilde{v} + \tilde{w} \star i\xi z). \tag{7}$$

Let $\|z(\xi,t)\| := u(t)$ and $\int_{\mathbb{R}^3} := \int$. From (7) and (6) one gets

$$u(t) \le c\int_0^t ds \int d\xi e^{-\nu\xi^2(t-s)}(1+|\xi|)\left[\int |z(\xi-\zeta,s)||\zeta||\tilde{v}(\zeta,s)|d\zeta + \int |\tilde{w}(\xi-\zeta,s)||\zeta||z(\zeta,s)|d\zeta\right]. \tag{8}$$

Let $\eta := \xi - \zeta$. One has:

$$\int d\zeta|\zeta||\tilde{v}| \int d\xi(1+|\xi|)|z(\xi-\zeta,s)|e^{-\nu\xi^2(t-s)} \le \|\tilde{v}\|u(s) \max_{\zeta\in\mathbb{R}^3}\left\{e^{-\nu|\eta+\zeta|^2(t-s)}\frac{1+|\eta+\zeta|}{1+|\eta|}\right\}. \tag{9}$$

Furthermore,

$$\max_{\zeta\in\mathbb{R}^3}\left\{e^{-\nu|\eta+\zeta|^2(t-s)}\frac{1+|\eta+\zeta|}{1+|\eta|}\right\} = (1+|\eta|)^{-1}\max_{p\in\mathbb{R}_+}\{(1+p)e^{-\nu p^2(t-s)}\} \le 1 + \frac{c_\nu}{(t-s)^{1/2}}, \tag{10}$$

where $c_\nu = c\nu^{-0.5}$. Indeed, if $h(r) = (1+r)e^{-\nu(t-s)r^2}$, then $\max_{r\ge 0} h(r) = h(R) \le 1 + \frac{c_\nu}{(t-s)^{1/2}}$, where $R = -\frac{1}{2} + (\frac{1}{4} + \frac{1}{2\nu(t-s)})^{1/2}$ and $h'(R) = 0$.

A similar estimate holds for the second integral in (8):

$$\int d\zeta(1+|\zeta|)|z(\zeta,s)| \int d\xi e^{-\nu\xi^2(t-s)}(1+|\xi|)|\tilde{w}(\xi-\zeta,s)| \le u(s)\max_{\zeta\in\mathbb{R}^3}\int dp|\tilde{w}(p,s)|(1+|p+\zeta|)e^{-\nu\xi^2(t-s)}. \tag{11}$$

The right side of (11) is $u(s)J$, where

$$J = \int dp|\tilde{w}(p,s)|(1+p)\max_{\zeta\in\mathbb{R}^3}\left(\frac{1+|p+\zeta|}{1+p}e^{-\nu\xi^2(t-s)}\right) \le \|\tilde{w}\|\left(1+\frac{c_\nu}{(t-s)^{1/2}}\right). \tag{12}$$

From (7)–(12) one gets

$$u(t) \le C(t)\int_0^t \left(1+\frac{c_\nu}{(t-s)^{1/2}}\right)u(s)ds, \quad C(t) = c\max_{0\le s\le t}(\|\tilde{v}(p,s)\| + \|\tilde{w}(p,s)\|), \tag{13}$$

where $C(t) > 0$ is a continuous function and $u(t) \ge 0$. Note that $C(t)$ is a continuous function of t for all $t \ge 0$ because we assume that the solutions $\tilde{v}$ and $\tilde{w}$ belong to X and $C(t)$ is the sum of the norms of the two elements of X. The Volterra inequality (13) has only the trivial solution $u(t) = 0$, as follows from Lemma 1, proved below. Theorem 1 is proved. □

Lemma 1. *Inequality (13) has only the trivial non-negative solution $u(t) = 0$.*

Proof of Lemma 1. Denote $\frac{u(t)}{C(t)} = q(t)$. Then

$$q(t) \leq \int_0^t \left(1 + \frac{c_\nu}{(t-s)^{1/2}}\right) C(s)q(s)ds := \int_0^t K(t,s)q(s)ds. \tag{14}$$

The kernel $K(t,s) > 0$ is weakly singular. Any solution $q \geq 0$ to (14) satisfies the estimate $0 \leq q \leq Q$, where $Q \geq 0$ solves the Volterra equation

$$Q(t) = \int_0^t K(t,s)Q(s)ds. \tag{15}$$

This equation has only the trivial solution $Q = 0$. Lemma 1 is proved. □

Proof of Theorem 2. From (5) after multiplying by $1 + |\xi|$, integrating over $\mathbb{R}^3$ and using calculations similar to the ones in equation (12), one gets

$$u(t) \leq b(t) + c\int_0^t \left(1 + \frac{c_\nu}{(t-s)^{1/2}}\right) u^2(s)ds := A(u), \tag{16}$$

where $b(t) := \int |\tilde{F}(\xi,t)|(1+|\xi|)d\xi$ and $u(t) := \|\tilde{v}(\xi,t)\|$. For sufficiently small T equation $U = AU$ is uniquely solvable by iterations according to the contraction mapping principle. If $\sup_{t\in[0,T]} b(t) \leq c_0$ and T is sufficiently small, then a ball $\sup_{t\in[0,T]} u(t) \leq c_1$, $c_1 > c_0$, is mapped by the operator A into itself and A is a contraction mapping. The operator A maps positive functions into positive functions. Thus, $u(t) \leq U(t)$. Theorem 2 is proved. □

Proof of Theorem 3. Under the assumption of Theorem 3 inequality (16) implies:

$$u(t) \leq b(t) + cc_a\int_0^t \left(1 + \frac{c_\nu}{(t-s)^{1/2}}\right) u(s)ds := A_1(u), \tag{17}$$

The corresponding equation $U = A_1U$ is a linear Volterra integral equation. It has a unique solution defined for all $t \geq 0$, and $0 \leq u(t) \leq U(t)$. Theorem 3 is proved. □

Remark 1. The following a priori estimates for solutions to NSP hold:

$$\|v\|_{L^2(\mathbb{R}^3)} \leq c, \quad \int_0^t \|\nabla v(x,s)\|^2_{L^2(\mathbb{R}^3)}ds \leq c, \tag{18}$$

and

$$\sup_{t\in[0,T]} |\tilde{v}(\xi,t)| \leq c + cT^{1/2}, \quad sup_{t\geq 0;\xi\in\mathbb{R}^3}(|\xi||\tilde{v}|) < c. \tag{19}$$

Proof of (18). First estimate (18) is well known. It remains to prove the second estimate (18). For this, multiply (1) by v and integrate over $\mathbb{R}^3$ to get (see [1]):

$$0.5\frac{d\int v^2dx}{dt} + \nu\int |\nabla v|^2dx = \int fvdx.$$

Integrating over t one gets:

$$0.5\int v^2dx + \int_0^t ds\int |\nabla v(x,s)|^2dx \leq 0.5\int v_0^2dx + \int_0^t ds\int fvdx.$$

One has $\int_0^t ds \int fv dx \le \int_0^t ds(\int |f(x,s)|^2)^{1/2}(\int |v(x,s)|^2)^{1/2} \le c$. Indeed, it is assumed that f decays fast, so $\sup_{t\ge 0}\int_0^t ds(\int |f|^2 dx)^{1/2} \le c$. Using this and estimates (18) we get $\int_0^t ds \int |fv| dx \le c$. Thus, the second estimate (18) is proved. □

Proof of estimate (19). From Equation (5) one gets:

$$|\tilde{v}| \le |\tilde{F}| + c\int_0^t e^{-\nu\xi^2(t-s)}|\tilde{v}| \star (|\xi||\tilde{v}|)ds := |\tilde{F}| + I. \tag{20}$$

One has $\sup_{t\ge 0}|\tilde{F}| \le c$ under the assumptions of Theorem 1. By the Cauchy inequality, the first estimate (18) and Parseval's equality one gets $|\tilde{v}| \star (|\xi||\tilde{v}) \le \|\tilde{v}|\|_{L^2(\mathbb{R}^3)}\||\xi|\tilde{v}\|_{L^2(\mathbb{R}^3)}$. Thus, using the Cauchy inequality, and the second estimate (18), one gets

$$I \le c\int_0^t e^{-\nu\xi^2(t-s)}\||\xi|\tilde{v}\|_{L^2(\mathbb{R}^3)}ds \le ct^{1/2}[\int_0^t \||\xi|\tilde{v}\|^2_{L^2(\mathbb{R}^3)}]^{1/2} \le ct^{1/2}. \tag{21}$$

From (20) and (21) estimate (19) follows.

The second estimate (19) is proved in [1], p. 50, inequality (5.39), under the assumption of Theorem 1. □

Conflicts of Interest: "The author declares no conflict of interest."

References

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