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On Hyperoctahedral Enumeration System, Application to Signed Permutations

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, we give the definitions and basic facts about hyperoctahedral number system. There is a natural correspondence between the integers expressed in the latter and the elements of the hyperoctahedral group when we use the inversion statistic on this group to code the signed permutations. We show that this correspondence provides a way with which the signed permutations group can be ordered. With this classification scheme, we can find the r-th signed permutation from a given number r and vice versa without consulting the list in lexicographical order of the elements of the signed permutations group.

Keywords: Hyperoctahedral enumeration system; signed permutation code; inversion statistic; lexicographic order.

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1 Introduction

The signed permutation groups, also known as the Weyl groups of type B or as the hyperoctahedral groups, are fundamental objects in today's mathematics. A better understanding of these groups may help to advance research in many fields like in cryptography [1] for instance, as well as in a number of other areas in discrete mathematics and combinatorics [2, 3]. One method of studying these groups is by using numerical statistics [4, 5, 6]. This method was successfully applied in the case of the symmetric groups [7, 8, 9, 10]. Thanks to the existence of the inversion statistic on the symmetric group, Laisant observes in a paper [11] that there is an ordering number corresponding to each permutation when the permutations are ordered lexicographically.

In this paper, we present how it is possible to generate the rank of each signed permutation when we classify signed permutations in lexicographic order. However for studying the proposed ordering, we introduce a new statistic, number of i-inversions, on the hyperoctahedral group with which we code the signed permutations. The first objective of this work will be to understand the hyperoctahedral enumeration system and to give some properties of the numbers in this system. The second objective would be to give, as application of the hyperoctahedral system, a classification in lexicographic order of signed permutations of elements of $\{1, \ldots, n\}$ when the elements of the set $\{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$ are ordered as follow : $1, 2, \cdots, n, -n, \cdots, -2, -1$.

1.1 Lexicographic ordering

The easiest way to explain lexicographic ordering is with an example.

Example 1.1. *The set of all permutations of order three in lexicographic order is:*

$$abc, acb, bac, bca, cab, cba.$$

1.2 Hyperoctahedral group

Let us denote by:

- $\mathbb{N}$ the set of non negative integers including 0,
- $[n]$ the set $\{1, \cdots, n\}$,
- $[\pm n]$ the set $\{-n, \cdots, -1, 1, \cdots, n\}$,
- $\mathcal{S}_n$ the symmetric group of degree n .

We represent an element σ of $\mathcal{S}_n$ as the word $\sigma_1 \cdots \sigma_n$ where $\sigma_i = \sigma(i)$.

Definition 1.1. A bijection $\pi : [\pm n] \longrightarrow [\pm n]$ satisfying $\pi(-i) = -\pi(i)$ for all i in $[\pm n]$ is called "signed permutation".

We also write a signed permutation π in the form

$$\pi = \begin{pmatrix} 1 & 2 & \ldots & n \\ \varepsilon_1\sigma_1 & \varepsilon_2\sigma_2 & \ldots & \varepsilon_n\sigma_n \end{pmatrix} \text{ with } \sigma \in \mathcal{S}_n \text{ and } \varepsilon_i \in \{\pm 1\} .$$

Under the ordinary composition of mappings, all signed permutations of the elements of $[n]$ form a group $\mathcal{B}_n$ called hyperoctahedral group of rank n.

1.3 Inversion statistic on $\mathcal{S}_n$ and number of i-inversions on $\mathcal{B}_n$

Definition 1.2. Let $\sigma \in \mathcal{S}_n$. A pair of indices (i, j) with $i < j$ and $\sigma_i > \sigma_j$ is called an inversion of σ.

Let $\{e_1, e_2, \ldots, e_n\}$ be the set of standard basis vectors of the vector space $\mathbb{R}^n$.

To define the new statistic number of i-inversions on $\mathcal{B}_n$, it is convenient to see $\mathcal{B}_n$ as the Coxeter group (see [6]) with root system

$$\Phi_n = \{\pm e_i,\ \pm e_i \pm e_j \mid 1 \leq i \neq j \leq n\},$$

and positive root system

$$\Phi_n^+ = \{e_k,\ e_i + e_j,\ e_i - e_j \mid k \in [n],\ 1 \leq i < j \leq n\}.$$

Let us consider the following subset of Φ_n^+ defined by

$$\Phi_{n,i}^+ = \{e_i,\ e_i + e_j,\ e_i - e_j \mid i < j \leq n\}. \tag{1.1}$$

Definition 1.3. We define the number of i-inversions of the signed permutation $\pi \in \mathcal{B}_n$ by

$$\texttt{inv}_i\,\pi = \#\{v \in \Phi_{n,i}^+ \mid \pi^{-1}(v) \in -\Phi_n^+\}. \tag{1.2}$$

Example 1.2. *In Table 1, we see the corresponding* 1*-inversions and* 2*-inversions of the eight elements of the hyperoctahedral group* $\mathcal{B}_2$.

Table 1. The 1-inversions and 2-inversions of the elements of $\mathcal{B}_2$

Signed permutation π	$\texttt{inv}_1\pi$: $\texttt{inv}_2\pi$
$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} \end{pmatrix}$	0 : 0
$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{1} & -\mathbf{2} \end{pmatrix}$	0 : 1
$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} \end{pmatrix}$	1 : 0
$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & -\mathbf{1} \end{pmatrix}$	1 : 1
$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ -\mathbf{2} & \mathbf{1} \end{pmatrix}$	2 : 0
$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ -\mathbf{2} & -\mathbf{1} \end{pmatrix}$	2 : 1
$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ -\mathbf{1} & \mathbf{2} \end{pmatrix}$	3 : 0
$\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ -\mathbf{1} & -\mathbf{2} \end{pmatrix}$	3 : 1

2 Hyperoctahedral Enumeration System

Let us consider the sequence of integers $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ with $B_i = 2^i i!$. This means that

$$\mathcal{B} = (1, 2, 8, 48, \cdots)\ .$$

Let us now take an integer $N > 0$, it is obvious that

$$B_n \leqslant N < B_{n+1} \text{ for some } n \in \mathbb{N} \,.$$

By dividing N by B_n, one obtains

$$N = d_n B_n + R_1 \text{ with } \begin{cases} 0 \leqslant R_1 < B_n \\ 1 \leqslant d_n \leqslant 2n+1 \end{cases} . \tag{2.1}$$

Here is why $1 \leqslant d_n \leqslant 2n+1$.

Proof. We have on the one hand $d_n \geqslant 1$ because $N \geqslant B_n$, on the other hand assume that

$$d_n \geqslant 2(n+1), \text{it follows that}$$
$$N \geqslant d_n B_n \geqslant 2(n+1)2^n n! = B_{n+1}$$

which is in contradiction with $N < B_{n+1}$, so we have $d_n < 2(n+1)$. □

Concerning the remainder R_1, one distinguishes two cases : $R_1 \neq 0$ and $R_1 = 0$. For $R_1 \neq 0$, we substitute N by it then, we divide it by B_{n-1}. In order to allow us to repeat the same operation on the next remainders, let us assume that each time we divide R_i by B_{n-i} we obtain $R_{i+1} \neq 0$ as remainder. This means that first of all, we have

$$R_1 = d_{n-1} B_{n-1} + R_2 \text{ with } \begin{cases} 0 < R_2 < B_{n-1} \\ 0 \leqslant d_{n-1} < 2n \end{cases} . \tag{2.2}$$

We have seen that $d_n \neq 0$ but here d_{n-1} may be zero. According to our hypothesis $R_1 \neq 0$, we just know from equation (2.1) that $0 < R_1 < B_n$. That's why, one or other of the following cases appears :

- $R_1 < B_{n-1}$ which gives $d_{n-1} = 0$
- $R_1 \geqslant B_{n-1}$, which gives $d_{n-1} \geqslant 1$.

Thus we obtain the first inequality $0 \leqslant d_{n-1}$. As $R_1 < B_n$, we have $d_{n-1} < 2n$. Therefore, $0 \leqslant d_{n-1} < 2n$. From equations (2.1) and (2.2), we write

$$N = d_n B_n + d_{n-1} B_{n-1} + R_2 \,.$$

By continuing in this way, we have for $i = 2, 3$ and so on

$$R_i = d_{n-i} B_{n-i} + R_{i+1} \text{ with } \begin{cases} 0 < R_{i+1} < B_{n-i} \\ 0 \leqslant d_{n-i} \leqslant 2(n-i)+1 \end{cases} .$$

At last, the integer N may be written in the form

$$N = d_n B_n + \cdots + d_1 B_1 + d_0 B_0 \text{ where } \begin{cases} d_i \in \{0, 1, 2, \cdots, 2i+1\} \\ B_i = 2^i i! \end{cases} . \tag{2.3}$$

By convention, we denote this integer N by the representation

$$d_n : d_{n-1} : d_{n-2} : \ \cdots \ : d_2 : d_1 : d_0 \text{ where the } d_i\text{'s are digits.}$$

Let us now deal with $R_1 = 0$. Throughout the successive divisions, if one of the obtained remainders is zero, then from this remainder, all the digits d_i's will be zero. Let $R_k = 0$ be this remainder , that is $R_{k-1} = d_{n-k+1} B_{n-k+1} + 0$ and the notation of the integer N will be

$$d_n : d_{n-1} : \ \cdots \ : d_{n-k+1} : \underbrace{0 : \ \cdots \ : 0}_{n-k+1 \text{ times}} \,.$$

For instance, if $N = d_n B_n + 0$, i.e $R_1 = 0$, we write $N = d_n : \underbrace{0 : \ \cdots \ : 0}_{n \text{ times}}$.

Like this, we have just written an integer $N > 0$ in a special enumeration system.

Definition 2.1. Hyperoctahedral number system is a system that expresses all natural number n of $\mathbb{N}$ in the form:

$$n = \sum_{i=0}^{k(n)} n_i.B_i \ , \text{ where } k(n) \in \mathbb{N},\ n_i \in \{0, 1, 2, \cdots, 2i+1\} \text{ and } B_i = 2^i i! \ . \tag{2.4}$$

This definition is motivated by the fact that we have taken $\mathcal{B} = (B_0, B_1, B_2, B_3, \cdots)$ as basis of the enumeration system where B_i is the cardinal of the hyperoctahedral group $\mathcal{B}_i$.

Definition 2.2. Let $n = d_{k-1} : d_{k-2} : \cdots : d_1 : d_0$ be a number in the hyperoctahedral system. We say that n is a k-digits number if the first digit d_{k-1} is not zero.

From equation (2.3) with the hypothesis $B_n \leqslant N < B_{n+1}$ on page 43, we see that any number between B_k and B_{k+1} is a $(k+1)$-digits number to the base $\mathcal{B} = (B_i)_{i\in\mathbb{N}}$.

Converting from one base to another Let us convert an integer from decimal system to hyperoctahedral system by means of Horner's procedure. Actually it is a scheme based on expressing a polynomial by a particular expression. For instance:

$$a + bx + cx^2dx^3 + ex^4 \ = \ a + x \cdot (b + x \cdot (c + x \cdot (d + x \cdot e)))\,.$$

To express a positive integer n in the hyperoctahedral system, one proceeds with the following manner. Start by dividing n by 2 and let d_0 be the rest r_0 of the expression

$$n = r_0 + (2)q_0 \ .$$

Divide q_0 by 4, and let d_1 be the rest r_1 of the expression

$$q_0 = r_1 + (4)q_1 \ .$$

Continue the procedure by dividing q_{i-1} by $2(i+1)$ and taking $d_i := r_i$ of the expression

$$q_{i-1} = r_i + 2(i+1)q_i$$

until $q_l = 0$ for some $l \in \mathbb{N}$. In this way, we obtain $n = d_l : d_{l-1} : \cdots : d_1 : d_0$ and we also have

$$n = d_0 + 2\,(d_1 + 4 \cdot (d_2 + 2(3) \cdot (d_3 + \cdots)))\,.$$

Now let $n = d_{k-1} : d_{k-2} : \cdots : d_1 : d_0$ be a number in the hyperoctahedral system. By definition 2.1, one way to convert n to the usual decimal system is to calculate

$$d_{k-1}2^{k-1}(k-1)! + \cdots + d_1.2 + d_0 \ .$$

In practice, one can use this algorithm:

Input :	An integer $d_{k-1} : d_{k-2} : \cdots : d_1 : d_0$ in the hyperoctahedral system.
Output :	An integer d in the decimal system.

1. **initiate the value of** d : $d \leftarrow d_{k-1}$
2. **for** i from $k-1$ to 1 do : $d \leftarrow d.2.i + d_{i-1}$
3. **return** d

Example 2.1. *To convert the number* $7:0:2:3:1$ *to the decimal system, multiply 1, 3, 2, 0, 7 respectively by* B_0, B_1, B_2, B_3, B_4, *after that, add the results :*

$$7(384) + 0 + 2(8) + 3(2) + 1 = 2711 \ .$$

We obtain the same result with :

$$\begin{aligned} d_4 = & \quad 7 \ , \\ 7(2)4 + d_3 = & \quad 56 + 0 = \quad 56 \ , \\ 56(2)3 + d_2 = & \quad 336 + 2 = \quad 338 \ , \\ 338(2)2 + d_1 = & \quad 1352 + 3 = \quad 1355 \ , \\ 1355(2)1 + d_0 = & \quad 2710 + 1 = \quad 2711 \ . \end{aligned}$$

Let us now convert 2711 *to the hyperoctahedral system by dividing it by* 2, *the obtained quotient by* 4, *and so on until we have zero as quotient. The digits that we search are the successive remainders. We shall find* $7:0:2:3:1$.

$$\begin{array}{r|r|r|r|r|r} 2711 & \underline{2} & & & & \\ 11 & 1355 & \underline{4} & & & \\ 11 & 15 & 338 & \underline{6} & & \\ 1 & 35 & 38 & 56 & \underline{8} & \\ & 3 & 2 & 0 & 7 & \underline{10} \\ & & & & 7 & 0 \end{array}$$

The first ninety numbers written in the hyperoctahedral system are given in Table 2.

Table 2. Representing positive integers in the decimal system and in the hyperoctahedral system

Decimal system	Hyperoctahedral system	Decimal system	Hyperoctahedral system	Decimal system	Hyperoctahedral system
0	0	30	330	60	1120
1	1	31	331	61	1121
2	10	32	400	62	1130
3	11	33	401	63	1131
4	20	34	410	64	1200
5	21	35	411	65	1201
6	30	36	420	66	1210
7	31	37	421	67	1211
8	100	38	430	68	1220
9	101	39	431	69	1221
10	110	40	500	70	1230
11	111	41	501	71	1231
12	120	42	510	72	1300
13	121	43	511	73	1301
14	130	44	520	74	1310
15	131	45	521	75	1311
16	200	46	530	76	1320
17	201	47	531	77	1321
18	210	48	1000	78	1330
19	211	49	1001	79	1331
20	220	50	1010	80	1400
21	221	51	1011	81	1401
22	230	52	1020	82	1410
23	231	53	1021	83	1411
24	300	54	1030	84	1420
25	301	55	1031	85	1421
26	310	56	1100	86	1430
27	311	57	1101	87	1431
28	320	58	1110	88	1500
29	321	59	1111	89	1501

3 Lexicographic Classification of Signed Permutations

Let $n \in \mathbb{N}$ and $\pi \in \mathcal{B}_n$. We code the signed permutation π by $\texttt{inv}_1\pi : \cdots : \texttt{inv}_n\pi$.

3.1 Signed permutation's rank

Lemma 3.1. *let $i, j \in [n]$ and $\pi \in \mathcal{B}_n$. If*

(i) $\pi(i) = j$, *then*

$$\texttt{inv}_i\,\pi = \#\{k \in \{i+1, \ldots, n\} \mid j > |\pi(k)|\},$$

(ii) $\pi(i) = -j$, *then*

$$\texttt{inv}_i\,\pi = 1 + \#\{k \in \{i+1, \ldots, n\} \mid j > |\pi(k)|\} + 2.\#\{k \in \{i+1, \ldots, n\} \mid j < |\pi(k)|\}.$$

Proof. Use the definition of the number of i-inversions (equations (1.1) and (1.2)). □

Lemma 3.2. *Let $i \in [n]$ and $\pi \in \mathcal{B}_n$. We have $\texttt{inv}_i\pi \in \{0, 1, \ldots, 2(n-i)+1\}$.*

Proof. We deduce that from Lemma 3.1. □

From lemma 3.2 we see that :

$$\begin{aligned}
&\texttt{inv}_1\pi \in \{0, 1, \ldots, 2n-1\},\\
&\texttt{inv}_2\pi \in \{0, 1, \ldots, 2(n-2)+1\},\\
&\vdots\\
&\texttt{inv}_{n-1}\pi \in \{0, 1, 2, 3\},\\
&\texttt{inv}_n\pi \in \{0, 1\}.
\end{aligned}$$

In other words, the code $\texttt{inv}_1\pi : \cdots : \texttt{inv}_n\pi$ has the same property than a n-digits number in the hyperoctahedral system. When we arrange in lexicographic order all the elements of the hyperoctahedral group $\mathcal{B}_n$, then the rank of π is $1+p$ where $\texttt{inv}_1\pi : \cdots : \texttt{inv}_n\pi$ represents the number p in the hyperoctahedral system.

Example 3.3. *Let us consider the signed permutation* $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -3 & 4 & 2 \end{pmatrix}$. *From equations (1.1) and (1.2),*

$$\texttt{inv}_1\pi : \texttt{inv}_2\pi : \texttt{inv}_3\pi : \texttt{inv}_4\pi \;=\; 0:4:1:0$$

which is the representation of $0(48) + 4(8) + 1(2) + 1(0) = 34$ *in the hyperoctahedral system. The rank of π is* $34 + 1 = 35$ *in the hyperoctahedral group $\mathcal{B}_4$.*

Given a signed permutation of the elements of $[n]$, $n > 0$, we have just seen a kind of classification with which we determine the rank of this permutation.

For instance, we give in Table 3 the classification of elements of the signed permutations group $\mathcal{B}_3$ of rank 3.

Table 3. Elements of the hyperoctahedral group $\mathcal{B}_3$ with the indication of their rank and their corresponding number in the hyperoctahedral system

Rank	$\pi = \pi_1\pi_2\pi_3$	$\mathtt{inv}_1\pi : \mathtt{inv}_2\pi : \mathtt{inv}_3\pi$	Rank	$\pi = \pi_1\pi_2\pi_3$	$\mathtt{inv}_1\pi : \mathtt{inv}_2\pi : \mathtt{inv}_3\pi$
1	1 2 3	0:0:0	25	-3 1 2	3:0:0
2	1 2-3	0:0:1	26	-3 1-2	3:0:1
3	1 3 2	0:1:0	27	-3 2 1	3:1:0
4	1 3-2	0:1:1	28	-3 2-1	3:1:1
5	1-3 2	0:2:0	29	-3-2 1	3:2:0
6	1-3-2	0:2:1	30	-3-2-1	3:2:1
7	1-2 3	0:3:0	31	-3-1 2	3:3:0
8	1-2-3	0:3:1	32	-3-1-2	3:3:1
9	2 1 3	1:0:0	33	-2 1 3	4:0:0
10	2 1-3	1:0:1	34	-2 1-3	4:0:1
11	2 3 1	1:1:0	35	-2 3 1	4:1:0
12	2 3-1	1:1:1	36	-2 3-1	4:1:1
13	2-3 1	1:2:0	37	-2-3 1	4:2:0
14	2-3-1	1:2:1	38	-2-3-1	4:2:1
15	2-1 3	1:3:0	39	-2-1 3	4:3:0
16	2-1-3	1:3:1	40	-2-1-3	4:3:1
17	3 1 2	2:0:0	41	-1 2 3	5:0:0
18	3 1-2	2:0:1	42	-1 2-3	5:0:1
19	3 2 1	2:1:0	43	-1 3 2	5:1:0
20	3 2-1	2:1:1	44	-1 3-2	5:1:1
21	3-2 1	2:2:0	45	-1-3 2	5:2:0
22	3-2-1	2:2:1	46	-1-3-2	5:2:1
23	3-1 2	2:3:0	47	-1-2 3	5:3:0
24	3-1-2	2:3:1	48	-1-2-3	5:3:1

3.2 Generating signed permutation in lexicographic order

Now, considering the rank k of a signed permutation, we want to generate the k-th signed permutation of the hyperoctahedral group $\mathcal{B}_n$. An efficient way to derive such signed permutation is to first convert $k-1$ in the hyperoctahedral system and then use the result to compute the corresponding permutation. Actually, each number in the hyperoctahedral system determines an unique signed permutation.

Let us denote $k-1$ by $\gamma_{n-1} : \cdots : \gamma_0$ in the hyperoctahedral system. Recall that we search the corresponding signed permutation of rank k. As the numbers of i-inversions have exactly the same property than the digits in the hyperoctahedral system, we are going to generate a permutation π such that

$$\mathtt{inv}_1\pi : \cdots : \mathtt{inv}_n\pi = \gamma_{n-1} : \cdots : \gamma_0 \,.$$

We start by defining for $\ell \in \mathbb{N}$ and $n > 0$ the following mapping :

$$\begin{array}{rccl} \mathcal{M}_\ell : & [2n-1] \cup \{0\} & \longrightarrow & [n-1] \cup \{0\} \times \{-1, 1\} \\ & \gamma & \longmapsto & \begin{cases} (\gamma, 1) & \text{if } \gamma \leq \ell \\ (1 + 2\ell - \gamma, -1) & \text{if } \gamma > \ell \end{cases} \end{array} .$$

We have already seen that

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \varepsilon_1\sigma_1 & \varepsilon_2\sigma_2 & \dots & \varepsilon_n\sigma_n \end{pmatrix} \text{ with } \sigma \in \mathcal{S}_n \text{ and } \varepsilon_i \in \{\pm 1\}$$

denotes a signed permutation of elements of $[n]$. Thus finding π comes back to find a permutation σ of $\mathcal{S}_n$ and $\varepsilon_i \in \{\pm 1\}$. Considering $\mathcal{M}_i(\gamma_i) = (m_i, \epsilon_i)$ for $i \in [n-1] \cup \{0\}$, we obtain the two sequences :

$$\epsilon = (\epsilon_{n-1}, \dots, \epsilon_0) \text{ and } m = (m_{n-1}, \dots, m_0) \,.$$

Taking $\varepsilon_i = \epsilon_{n-i}$, we obtain the ε_i's. The next step to do is to find $\sigma \in \mathcal{S}_n$. By the definition of the mapping $\mathcal{M}_\ell$, we deduce that $m_i \in \{0, 1, 2, 3, \ldots, i\}$. Thereby

$$\begin{aligned}&m_{n-1} \in \{0, 1, 2, 3, \ldots, n-1\},\\&\vdots\\&m_1 \in \{0, 1\},\\&m_0 = 0\ .\end{aligned}$$

For a permutation of the symmetric group $\mathcal{S}_n$, the number of inversions between an object and those after the latter only varies from zero to p, where p indicates the number of the following objects. The m_i's have the same property than this number of inversions. Therefore, we are going to search $\sigma \in \mathcal{S}_n$ which verifies

$$\mathsf{inv}_1\sigma \ \cdots \ \mathsf{inv}_n\sigma = m_{n-1} \ \cdots \ m_0 \text{ where } \mathsf{inv}_i\sigma = \#\{i < j < n \mid \sigma(i) > \sigma(j)\}\ .$$

Let $r_i = 1 + m_{i-1}$.

σ_1 is the r_n-th element of the list : $1, 2, 3, \ldots, n$ and then one deletes it from the list. Then, σ_2 is the r_{n-1}-th element among the rest of the list and one also deletes it from this one. And so on σ_n is the unique element of the last list. This procedure can be found for instance in the work of Laisant [11].

Example 3.4. *One asks the 35-th signed permutation of the elements of the set* $[4]$. *In the hyperoctahedral system, we represent* $35 - 1 = 34$ *by* $0 : 4 : 1 : 0$ *(we take four digits because the set* $[4]$ *has four elements). We use the mappings* $\mathcal{M}_1, \ldots, \mathcal{M}_4$ *to obtain the two sequences :*

$$\epsilon = (1, -1, 1, 1) \text{ and } m = (0, 1, 1, 0)\ .$$

Adding an unit to each element of $m = (0, 1, 1, 0)$ *gives the ranks* $r_4 = 1,\ r_3 = 2,\ r_2 = 2,\ r_1 = 1$. *Thereby among the elements of the set* $[4]$ *: that is* $1, 2, 3, 4$ *written in this order, one takes the one of rank* $r_4 = 1$*, that is* 1*, then the second among* $2, 3, 4$ *which is* 3*, next one takes* 4 *or the one of rank* r_2 *among* $2, 4$*, at last the first of the list which is* 2*. Thus one has*

$$\sigma = 1342 \in \mathcal{S}_4\ .$$

From the sequence ϵ*, one forms the thirty fifth signed permutation of the hyperoctahedral group* $\mathcal{B}_4$ *:*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ (1)\sigma_1 & (-1)\sigma_2 & (1)\sigma_3 & (1)\sigma_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -3 & 4 & 2 \end{pmatrix}\ .$$

4 Conclusion

Writing integer in the hyperoctahedral enumeration system is a fundamental step to do when one wants to construct a one-to-one correspondence between natural integers of the set $\{1, \ldots, 2^n n!\}$ and signed permutations of the hyperoctahedral group $\mathcal{B}_n$. There are several ways to continue the next step which consists to associate bijectively a signed permutation to each number in the hyperoctahedral system. For example, one can use the so-called subexceedant function (see [1]) but the inconvenient is that this way does not provide lexicographic order of signed permutation. In this article, the use of the new statistic number of i-inversions of signed permutation has been introduced to define our bijection. This method is more advantageous. It allows to represent each signed permutation by an unique integer which also denotes the rank of this signed permutation in lexicographic order. The proposed bijection in this work may also have applications in other areas. For instance, one can use it for implementation of the hyperoctahedral group cryptography.

Competing Interests

Author has declared that no competing interests exist.

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