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Alpha Power Transformed Extended Bur II Distribution: Properties and Applications

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Authors' contributions

This work was carried out in collaboration among all authors. Author AAO designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors BA and DOO managed the analyses of the study. Author AAO managed the literature searches. All authors read and approved the final manuscript.

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Abstract

This paper presents a new generalization of the extended Bur II distribution. We redefined the Bur II distribution using the Alpha Power Transformation (APT) to obtain a new distribution called the Alpha Power Transformed Extended Bur II distribution. We derived several mathematical properties for the new model which includes moments, moment generating function, order statistics, entropy etc. and used a maximum likelihood estimation method to obtain the parameters of the distribution. Two real-world data sets were used for applications in order to illustrate the usefulness of the new distribution.

Keywords: Alpha power transformation; moments; order statistics; Bur II distribution; real data sets.

1 Introduction

Burr [1], introduced a system of distributions which contains the Burr XII (BXII) distribution as the most commonly used among different class of these distributions. If a random variable has the BXII distribution,

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it is such that X^{-1} has the scaled Burr III (BIII) distribution with cumulative distribution function (cdf) defined (for $X > 0$) by [2].

$$\bar{G}(x) = (1 + \left(\frac{x}{\sigma}\right)^{-\lambda})^{-\theta} \tag{1}$$

Where $x \in (0, \infty)$ and λ, σ, θ are real-valued parameters that determine the mean, variance, kurtosis, and skewness of a distribution.

The Burr Type III and Type XII distributions attract special attention because they include several families of nonnormal probability distributions (e.g., the Gamma distribution) with varying degrees of skewness and kurtosis [3,4,5,6]. Further, these distributions have been used in a variety of applied mathematics contexts. Some examples include modeling events associated with software reliability growth [7], risk measurement [8,9], life testing [10,11], forestry [12,13], modeling crop prices [14], reliability analysis [15], meteorology [16], fracture roughness [17,18] etc.

Suppose we let $\sigma = 1$ in (1) we obtain another unique distribution named Bur II distribution which can also be used effectively in modeling reliability problems, meteorology, life testing etc.

The cdf of Bur II distribution is given by

$$G(x) = (1 + x^{-\lambda})^{-\theta} \tag{2}$$

And the corresponding probability density function (pdf) is given by

$$g(x) = \lambda\theta x^{-(\lambda+1)}(1 + x^{-\lambda})^{-(\theta+1)} \tag{3}$$

Where $x \in (0, \infty)$ and λ, θ are real-valued parameters that determine the mean, variance, kurtosis, and skewness of the distribution.

1.1 Alpha power transformed extended Bur II (APTEBII) distribution

In recent time several methods have been developed to induce flexibility into standard probability distributions in order to increase their areas of applications and also to obtain a better fits. This work focuses on the use of the method developed by [19], called the Alpha Power Transformation (APT) to obtain a new distribution named Alpha Power Extended Bur II (APTEBII) distribution. [20] Studied the properties of Alpha Power extended Exponential distribution, [21] studied the properties of Alpha power transformed generalized exponential distribution. Alpha Power transformed Weibull distribution was investigated by [22].

Let $g(x) and\ G(x)$ represent the pdf and the cdf of a continuous random variable X, respectively. The APT of $G(x)$ for $x\epsilon\ R$ is given by

$$F(x)\begin{Bmatrix} \dfrac{\alpha^{G(x)} - 1}{\alpha - 1}, & if\ \alpha > 0, \alpha \neq 1 \\ G(x), & if\ \alpha = 1 \end{Bmatrix} \tag{4}$$

And the associated pdf is given by

$$f(x) = \begin{Bmatrix} \dfrac{log\alpha}{\alpha - 1} g(x)\alpha^{G(x)}, & if\ \alpha > 0, \alpha \neq 1 \\ g(x), & if\ \alpha = 1 \end{Bmatrix} \tag{5}$$

Putting (2) in (4), we obtain the cdf of Alpha Power Transformed Extended Bur II (APTEBII) distribution given by

$$F(x) = \begin{cases} \dfrac{\alpha^{(1+x^{-\lambda})^{-\theta}} - 1}{\alpha - 1}, & if\ \alpha > 0, \alpha \neq 1 \\ (1+x^{-\lambda})^{-\theta}, & if\ \alpha = 1 \end{cases} \tag{6}$$

The graph of the cumulative density function of APTEBII distribution is drawn below with the value of $\alpha = 10.5$

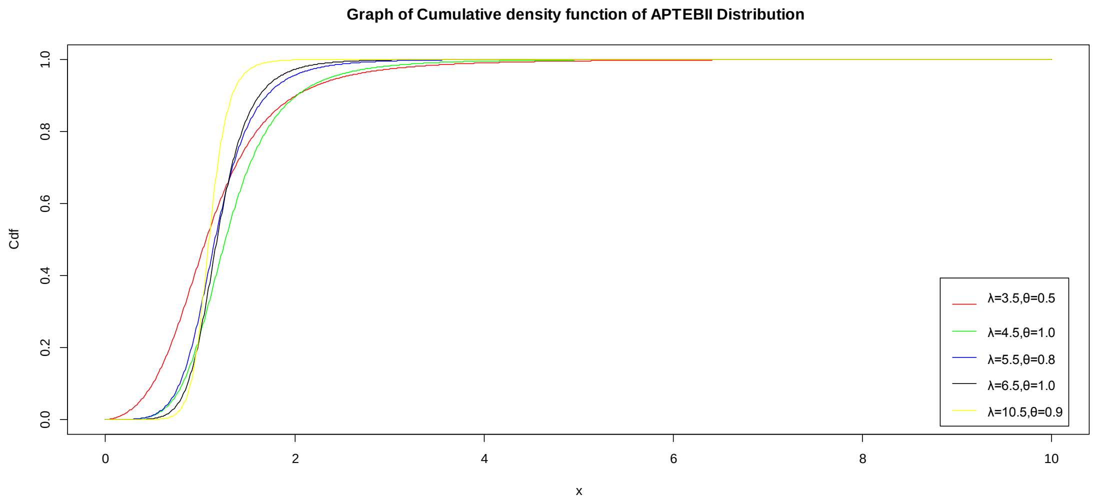


Fig. 1. The graph of the cdf of APTEBII distribution

The Fig. 1. drawn above indicates that APTEBII distribution has a proper pdf

And the pdf of APTEBII distribution is given by

$$f(x) = \begin{cases} \dfrac{\lambda\theta log\alpha}{\alpha - 1} x^{-(\lambda+1)}(1+x^{-\lambda})^{-(\theta+1)}\alpha^{(1+x^{-\lambda})^{-\theta}}, & if\ \alpha > 0, \alpha \neq 1 \\ \lambda\theta x^{-(\lambda+1)}(1+x^{-\lambda})^{-\theta}, & if\ \alpha = 1 \end{cases} \tag{7}$$

The graph of the pdf of APTEBII distribution is drawn below with the value of $\alpha = 0.5$

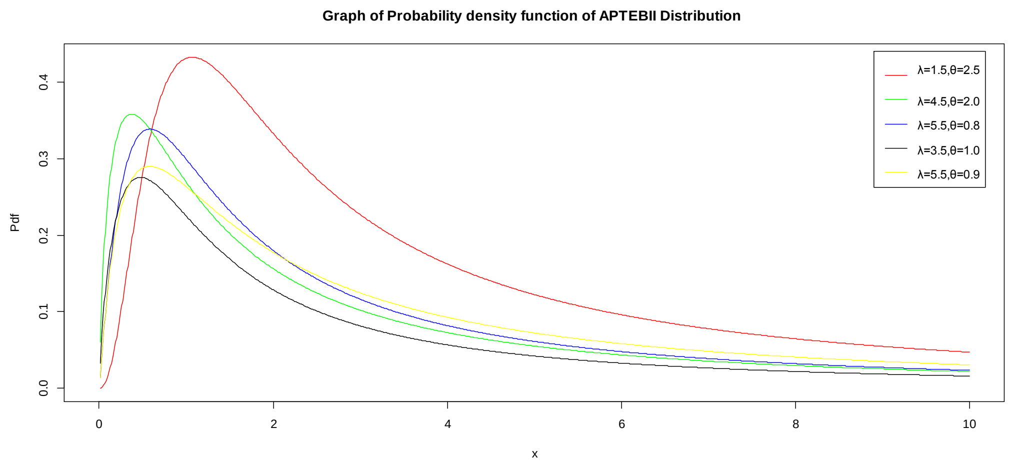


Fig. 2. The graph of the pdf of APTEBII distribution

The graph drawn above indicates that the APTEBII distribution is unimodal and can be used to address the problem of non-monotone failure rate that is common in real life data.

The survival function of APTEBII distribution is given by

$$S(x) = \begin{cases} \frac{\alpha}{\alpha-1}\left(1-\alpha^{(1+x^{-\lambda})^{-\theta}-1}\right), & if\ \alpha \neq 1 \\ 1-(1+x^{-\lambda})^{-\theta}, & if\ \alpha = 1 \end{cases} \tag{8}$$

The graph of the APTEBII distribution is drawn below with the value of taken as $\alpha = 0.5$

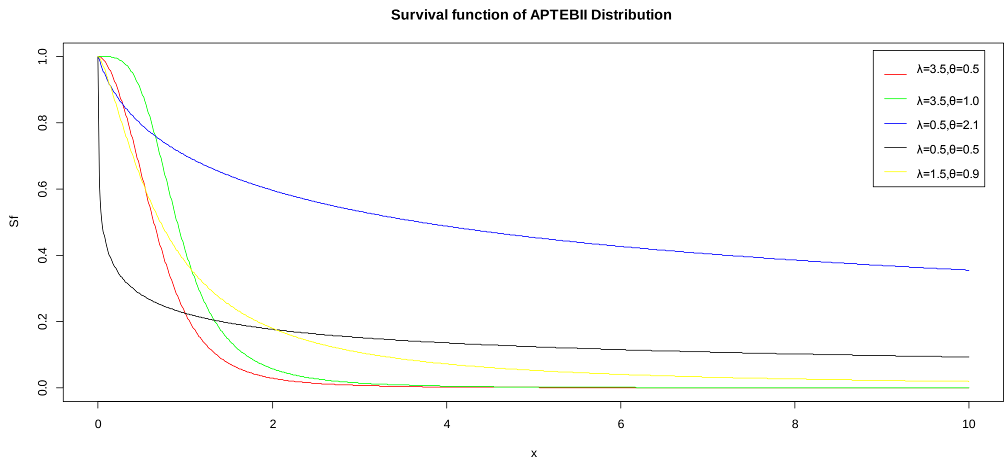


Fig. 3. The graph of the survival function of APTEBII distribution

And the hazard function is given by

$$h(x) = \begin{cases} \lambda\theta x^{-(\lambda+1)}(1+x^{-\lambda})^{-(\theta+1)}\dfrac{\alpha^{(1+x^{-\lambda})^{-\theta}-1}}{1-\alpha^{(1+x{-}^{-\lambda})^{-\theta}-1}}log\alpha, & if\ \ \alpha \neq 1 \\ \dfrac{\lambda\theta x^{-(\lambda+1)}(1+x^{-\lambda})^{-(\theta+1)}}{1-(1+x^{-\lambda})^{-\theta}}, & if\ \ \alpha = 1 \end{cases} \tag{9}$$

The graph of the hazard function of APTEBII distribution is drawn below with the values $\alpha = 0.5\ and\ \lambda = 1.5$

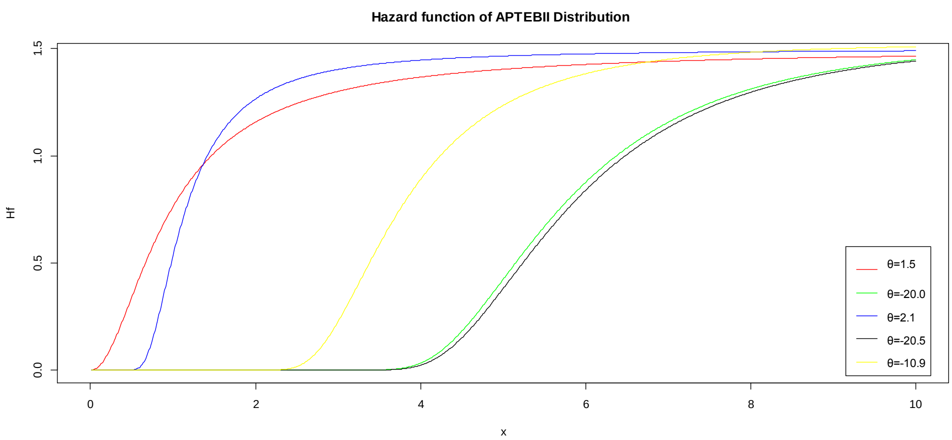


Fig. 4. The graph of the hazard function of APTEBII distribution

The graph drawn above indicates that the hazard function of APTEBII distribution is increasing.

2 Statistical Properties of APTEBII Distribution

The APTEBII distribution can be simulated by inverting cdf (6) as follows: if u follows uniform distribution on (0, 1), then

$$q(u) = \left[\left(\frac{1+u(\alpha-1)}{log\alpha}\right)^{-\frac{1}{\theta}} - 1\right]^{-\frac{1}{\lambda}}, 0 \leq u \leq 1. \tag{10}$$

In particular the first three quantile, q_1, q_2, q_3 for the APTEBII distribution were obtained by setting $u = 0.25$ for q_1, $u = 0.5$ for q_2 and $u = 0.75$ for q_3 representing the lower, middle and the upper quartiles.

2.1 Skewness and Kurtosis

The symmetry of a distribution is measured by the coefficient of skewness of a distribution and the coefficient of kurtosis is also a measure of the heaviness of the tail of the distribution.

The Bowley' skewness [23], is based on quartiles of a distribution as follows:

$$S = \frac{q\left(\frac{3}{4}\right) + q\left(\frac{1}{4}\right) - 2q\left(\frac{1}{2}\right)}{q\left(\frac{3}{4}\right) - q\left(\frac{1}{4}\right)} \tag{11}$$

And the Moors 'kurtosis [24] is based on octiles of a distribution, given by

$$\kappa = \frac{q\left(\frac{7}{8}\right) - q\left(\frac{5}{8}\right) - q\left(\frac{3}{8}\right) + q\left(\frac{1}{8}\right)}{q\left(\frac{6}{8}\right) - q\left(\frac{2}{8}\right)} \tag{12}$$

Where $q(.)$ represent the quantile function, and can be obtained from (10)

Table 1 displays the percentage points of some specific choices of the parameters taken ($\lambda = 2.5, \theta = 5.0$) and varying the value of parameter α. It contains the lower quartile ($u = 0.25$), median $u = 0.5$) and the upper quartile ($u = 0.75$), Bowley' skewness (s) and Moors 'kurtosis.

Table 1. Skewness and Kurtosis of the APTEBII distribution for different values of parameters

α	$\mathbf{0.25}$	$\mathbf{0.5}$	**0.75**	**1/8**	**3/8**	**5/8**	**7/8**	S	k
1.5	0.0335	0.0235	0.0171	0.0405	0.0279	0.0191	0.01474	-0.2152	-0.4511
2.5	0.2839	0.1188	0.0612	0.5214	0.0177	0.0837	0.04510	-0.4824	-2.0918
15	0.1943	0.0271	0.0081	5.3901	0.0622	0.0131	0.00503	-0.7956	-15.4411
20	0.1168	0.0153	0.0044	1.1462	0.0361	0.0077	0.00272	-0.8060	-9.8323
25	0.0761	0.0095	0.0027	0.6378	0.0229	0.0048	0.00165	-0.8135	-8.3333

The skewness, kurtosis, median, lower quartile and the upper quartile of the APTEBII distribution for several values of parameters are listed in Table 1. The values indicate that skewness and kurtosis are negative for all values of the parameters considered. This indicates that the distribution is negatively skewed and increased for the increase in the values α keeping the values of λ and θ fixed at 2.5 and 1.5 respectively.

Table 2 displays the percentage points of some specific choices of the parameters taken ($\lambda = 2.5, \theta = -1.5$) and varying the value of parameter α. It contains the lower quartile ($u = 0.25$), median ($u = 0.5$) and the upper quartile ($u = 0.75$), Bowley' skewness (s) and Moors 'kurtosis.

Table 2. Skewness and Kurtosis of the APTEBII distribution for different values of parameters

α	0.25	0.5	0.75	1/8	3/8	5/8	7/8	S	κ
1.5	29.8926	42.5415	58.5038	24.6736	35.8306	50.0807	67.8678	0.1158	0.2317
2.5	3.5219	8.4151	16.3457	1.9179	5.6469	11.9425	21.7430	0.2369	0.4734
15	5.1450	36.8826	123.954	0.1855	16.0801	71.4519	198.805	0.4657	0.9381
20	8.5615	65.3241	226.8845	0.8725	27.6963	129.008	367.603	0.4800	0.9700
25	13.1411	104.8401	371.6415	1.5679	43.6410	209.5513	605.8644	0.4884	0.9881

The skewness, kurtosis, median, lower quartile and the upper quartile of the APTEBII distribution for several values of parameters are listed in Table 2. The values indicate that skewness and kurtosis are positive for all values of the parameters and increased for the increase in the values α keeping the values of λ and θ fixed at 2.5 and -1.5 respectively. The APTEBII distribution is positively skewed and leptokurtic for all values of parameters considered.

2.2 Random number generation

The random variate Q from (APTEBII) distribution can be generated as q_u according to (10), where $u \sim U(0,1)$.

2.3 Moments

In this subsection, the r^{th} moment and moment generating function (mgf) of APTEBII distribution are derived. Using the following series representation in pdf (7)

$$\alpha^k = \sum_{p=0}^{\infty} \frac{(log\alpha)^p}{p!} k^p \tag{13}$$

Using the above expression, we can re-write the pdf in (7) as

$$f(x) = \frac{\lambda\theta}{\alpha - 1} \sum_{p=0}^{\infty} \frac{(log\alpha)^p}{p!} x^{-(\lambda+1)} (1 + x^{-\lambda})^{-[\theta(p+1)+1]} \tag{14}$$

And an expression for the r^{th} moment is given by

$$E(X^r) = \frac{\lambda\theta}{\alpha - 1} \sum_{p=0}^{\infty} \frac{(log\alpha)^p}{p!} \int_{-\infty}^{\infty} x^{r-(\lambda+1)} (1 + x^{-\lambda})^{-[\theta(p+1)+1]} \, dx \tag{15}$$

Letting, m=[$\theta(p + 1) + 1$] and $Z = \left(1 + x^{-\lambda}\right)$ in (15), we have

$$E(X^r) = \frac{\theta}{m(\alpha - 1)} \sum_{p=0}^{\infty} \frac{(log\alpha)^p}{p!} \int_{-\infty}^{\infty} Z^{-\frac{1}{m}} \left(Z^{-\frac{1}{m}} - 1\right)^{-\frac{r}{\lambda}} dz \tag{16}$$

Also, let $u = Z^{-\frac{1}{m}}$, then (16) will transform to

$$E(X^r) = \frac{\theta}{(\alpha-1)} \sum_{p=0}^{\infty} \frac{(log\alpha)^p}{p!} \int_{-\infty}^{\infty} Z^{-\frac{1}{m}} \left(Z^{-\frac{1}{m}} - 1\right)^{-\frac{r}{\lambda}} dz \qquad (17)$$

Substitute for $U = Z^{-\frac{1}{m}}$ in (17), then we have

$$E(X^r) = \frac{-\theta}{(\alpha-1)} \sum_{p=0}^{\infty} \frac{(log\alpha)^p}{p!} \int_{-\infty}^{\infty} U^{m+2}(U-1)^{-\frac{r}{\lambda}} dz \qquad (18)$$

Since,

$$-\int_{0}^{1} U^{m+2}(U-1)^{-\frac{r}{\lambda}} dz = \int_{-\infty}^{\infty} U^{m+2}(1-U)^{-\frac{r}{\lambda}} dz$$

Finally we have

$$E(X^r) = \mu_r' = \frac{\lambda}{\alpha-1} \sum_{p=0}^{\infty} \frac{(log\alpha)^p}{p!} B\left\{[\theta(p+1)+4], 1-\frac{r}{\lambda}\right\}, \quad \lambda > r \qquad (19)$$

Where

$$B(y,z) = \int_{0}^{1} x^{y-1} (1-x)^{z-1} dx$$

The first four moments can be obtained by setting $r = 1,2,3$ and 4 in (19). Also the r^{th} central moments can be obtained using the expression

$$\mu_r = E(X-\mu_1')^n = \sum_{r=0}^{n} \binom{n}{r} (-\mu_1')^{n-r} E(X^r) \qquad (20)$$

Thus the n^{th} central moment of APTEBII distribution is given by

$$\mu_r = \sum_{r=0}^{n} \sum_{p=0}^{\infty} \frac{(log\alpha)^p}{p!} \binom{n}{r} (-\mu_1')^{n-r} \frac{\lambda}{\alpha-1} B\left\{[\theta(p+1)+4], 1-\frac{r}{\lambda}\right\} \qquad (21)$$

It then follows that from (20), we can obtain the skewness (S) and kurtosis (k) define by

$$sk = \frac{\mu_3}{\mu_2^{\frac{3}{2}}}, \quad ku = \frac{\mu_4}{\mu_2^2}$$

Table 3. The mean and variance of APTEBII distribution

$\boldsymbol{\alpha, \lambda, \theta}$	$\boldsymbol{\mu_1'}$	$\boldsymbol{\mu_2'}$	$\boldsymbol{\mu_3'}$	$\boldsymbol{\mu_4'}$	$\boldsymbol{Var}$
5.0,1.5,5.5	0.0713	0.1374	0.4383	2.8919	0.1323
10.0,1.5,5.5	0.0409	0.1834	0.0695	0.0420	O.1817
10.0,2.5,8.5	0.0081	0.0012	0.0180	0.0324	0.0011
12.5,5.5,10.5	0.0409	0.0627	0.0999	0.1654	0.0610

2.4 Moment generating function of APTEBII distribution

Table 3 contains values of mean (μ_1'), variance(σ^2) and the coefficient of variation (CV) of APTEBII distribution for some certain values of parameters.

In this sub-section we derived the moment generating function of the APTEBII distribution.

The moment generating function of a distribution defined by

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \tag{22}$$

Putting (20) in (22), we have an expression for the moment generating function of APTEBII distribution given by

$$M_X(t) = \frac{\lambda}{\alpha - 1} \sum_{p=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{(log\alpha)^p}{p!} B\left\{[\theta(p+1)+4], 1 - \frac{r}{\lambda}\right\} \tag{23}$$

3 Renyl Entropy

The entropy of a random variable provides the basis for estimating the amount of information (or uncertainty) contained in a random observation in preference to its parent distribution (population). A large value of entropy implies the greater uncertainty in the data, [25]. The concept of entropy found applications in various field of life such as in science, engineering and economics. The Renyi entropy of a random variable ´ X, for $\Lambda \neq 0$ and $\Lambda \neq 1$, is defined by

$$\boldsymbol{I_{\Lambda}(X) = (1 - \Lambda)^{-1} log}\left(\int_{-\infty}^{\infty} [\boldsymbol{f(x)}]^{\Lambda}\right) \tag{24}$$

The Renyl entropy of APTEBII distribution can be obtained by inserting (7) in (24) as follows

$$I_{\Lambda}(X) = (1 - \Lambda)^{-1} log\left[\left\{\frac{log\alpha}{1-\alpha}\right\}^{\Lambda} \{\lambda\theta\}^{\Lambda} \int_{-\infty}^{\infty} x^{-\Lambda(\lambda+1)} (1 + x^{-\lambda})^{-\Lambda(\theta+1)} \alpha^{(1+x^{-\lambda})^{-\Lambda\theta}} dx\right] \tag{25}$$

By making appropriate substitution, the Renyl entropy of APTEBII distribution is given by

$$I_{\Lambda}(X) = -(1 - \Lambda)^{-1} log\left[\left\{\frac{log\alpha}{1-\alpha}\right\}^{\Lambda} \theta^{\Lambda} \lambda^{\Lambda - 1} \sum_{i=0}^{\infty} \frac{(log\alpha)^i}{i!} B\left[1 - \Lambda\{\theta(i+1)+1\}, \frac{\Lambda(\lambda+1)-1}{\lambda}\right]\right] \tag{26}$$

Taking the value of $\alpha = 1.5, \lambda = 1.5, \theta = 0.5$ and $\Lambda = 5$ the Renyl entropy was observed to be -0.6029

4 Order Statistics

The order statistics found applications in reliability and life testing and it also plays an important role in statistical inference.

4.1 Some distribution of order statistics

Suppose we let $X_{(1)} \leq X_{(2)} \leq ...X_{(n)}$ be an ordered observation in a random sample of size n drawn from APTEBII distribution with cdf, $F(x)$ given by (6) and pdf $f(x)$, given by (7). The pdf of $X_{(i)}$, $i = 1,2, ... n$ is given by

$$f_{X_{(i)}} = \frac{n!}{(i-1)!\,(n-i)!} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i}$$

$$f_{X_{(i)}} = \frac{n!}{(i-1)!\,(n-i)!} \frac{\lambda\theta log\alpha}{(\alpha-1)^n} x^{-(\lambda+1)} h^{-(\theta+1)} \alpha^{h^{-\theta}} \left[\alpha^{h^{-\theta}} - 1\right]^{i-1} \left[\alpha - \alpha^{h^{-\theta}}\right]^{n-i} \quad (27)$$

Where, $h = 1 + x^{-\lambda}$

Then the pdf of the largest order statistics $X_{(n)}(i = n)$, the smallest order statistics $X_{(1)}$ $(i = 1)$ and the distribution of the median order $X_{(m+1)}, when\ n = 2m + 1$ $(n\ is\ odd\ number)$ are respectively given by

$$f_{X_{(n)}} = n\lambda\theta(\alpha-1)^{-n}(log\alpha)x^{-(\lambda+1)}h^{-(\theta+1)}\alpha^{h^{-\theta}}\left[\alpha^{h^{-\theta}} - 1\right]^{n-1} \quad (28)$$

$$f_{X_{(1)}} = n\lambda\theta(\alpha-1)^{-n}x^{-(\lambda+1)}h^{-(\theta+1)}\alpha^{h^{-\theta}}\left[\alpha - \alpha^{h^{-\theta}}\right]^{n-1} \quad (29)$$

And

$$f_{X_{(2m+1)}} = \frac{(2m+1)!}{\left((m)!\right)^2} \frac{\lambda\theta log\alpha}{(\alpha-1)^n} x^{-(\lambda+1)} h^{-(\theta+1)} \alpha^{h^{-\theta}} \left[\left\{\alpha^{h^{-\theta}} - 1\right\}\left\{\alpha - \alpha^{h^{-\theta}}\right\}\right]^m \quad (30)$$

4.2 Joint distribution of i^{th} and j^{th} order statistics

The joint pdf of two order statistics (X_i, X_j) for $i, j = 1,2, . . . , n$ of the APTEBII distribution is given by

$$f_{\left(X_{(i)},X_{(j)}\right)}(t_i, t_j) = \mathcal{Z}\left[\frac{\alpha^{h_i^{-\theta}} - 1}{\alpha - 1}\right]^{i-1}\left[\frac{\alpha^{h_j^{-\theta}} - \alpha^{h_i^{-\theta}}}{\alpha - 1}\right]^{j-i-1}\left[\frac{\alpha - \alpha^{h_j^{-\theta}}}{\alpha - 1}\right]^{n-j}$$

$$\times\left\{\frac{\lambda\theta log\alpha}{\alpha-1} x_i^{-(\lambda+1)} h_i^{-(\theta+1)} \alpha^{h_i^{-\theta}}\right\}\left\{\frac{\lambda\theta log\alpha}{\alpha-1} x_j^{-(\lambda+1)} h_j^{-(\theta+1)} \alpha^{h_j^{-\theta}}\right\} \quad (31)$$

Where $\mathcal{Z} = \frac{n!}{(i-1)!(j-i-1)!(n-i)!}$, $h_i = 1 + x_i^{-\lambda}\ and\ h_j = 1 + x_j^{-\lambda}$

5 Estimation of Parameters

Here we derive the maximum likelihood estimators (MLEs) for the parameters of the APTEBII distribution. Let $(X_1, X_2, . . . X_n)$ be a random sample of size n from an APTEBII $(\alpha, \lambda, \theta)$ distribution then the likelihood function can be written as

$$L = \prod_{i=1}^{n} \frac{\lambda\theta log\alpha}{\alpha-1} x^{-(\lambda+1)}(1 + x^{-\lambda})^{-(\theta+1)}\alpha^{(1+x^{-\lambda})^{-\theta}} \quad (32)$$

Then, the log-likelihood function, l, is given by

$$l = n\left[\log(\alpha) + \log(\theta) + log\left(\frac{log\alpha}{\alpha-1}\right)\right] + log\alpha\sum_{i=1}^{n}(1 + x_i^{-\lambda})^{-\theta} - (\lambda+1)\sum_{i=1}^{n} log\, x_i - (\theta+1)\sum_{i=1}^{n} log\left(1 + x_i^{-\lambda}\right) \quad (33)$$

The log-likelihood function can be maximized either directly or by solving the nonlinear equations obtained by differentiating (33). Thus the components of the score vector are given by

$$\frac{\partial l}{\partial \alpha}=\frac{n(\alpha-1-\alpha log\alpha)}{\alpha(\alpha-1)log\alpha}+\frac{1}{\alpha}\sum_{i=1}^{n}(1+x_i{}^{-\lambda})^{-\theta} \tag{34}$$

$$\frac{\partial l}{\partial \lambda}=\lambda\theta\log(\alpha)\sum_{i=1}^{n}\frac{(1+x_i{}^{-\lambda})^{-\theta}x_i{}^{-\lambda}\log(x))}{1-x_i{}^{-\lambda}}+(\theta+1)\sum_{i=1}^{n}\frac{\lambda x_i{}^{-\lambda}\log(x)}{1+x_i{}^{-\lambda}} \tag{35}$$

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}-\sum_{i=1}^{n}(1+x_i{}^{-\lambda})^{-\theta}log\left(1-x_i{}^{-\lambda}\right)\log(\alpha)-\sum_{i=1}^{n}\log\left(1-x_i{}^{-\lambda}\right) \tag{36}$$

(34), (35) and (36) cannot be solved analytically. Therefore, statistical software such as R, MATLAB etc can be employed in obtaining the numerical solution to the non-linear equations. For the three parameter Alpha power transform extended Bur II distribution pdf, all the second order derivatives can be obtained. Thus the distribution of the random vector is given by

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\lambda}\\ \hat{\theta}\end{pmatrix}\sim N\left[\begin{pmatrix}\alpha\\ \lambda\\ \theta\end{pmatrix}\begin{pmatrix}\hat{J}_{\alpha\alpha} & \hat{J}_{\alpha\lambda} & \hat{J}_{\alpha\theta}\\ \hat{J}_{\lambda\alpha} & \hat{J}_{\lambda\lambda} & \hat{J}_{\lambda\theta}\\ \hat{J}_{\theta\alpha} & \hat{J}_{\theta\lambda} & \hat{J}_{\theta\theta}\end{pmatrix}\right],$$

Where $\hat{J}_{ij}=J_{ij}\big|_{v=v_i}=(\alpha,\lambda,\theta)$ with $\{J_{ij}\}=\left[-l_{ij}\right]^{-1}=\left[\frac{\partial^2 l}{\partial v_i \partial v_j}\right]$. This gives the approximate variance covariance matrix. By solving for the inverse of the dispersion matrix, the solution will give the asymptotic variance and covariance of the MLs for $\hat{\alpha}$, $\hat{\lambda}$, and $\hat{\theta}$. The approximate $100(1-c)\%$ confidence intervals for $\alpha, \lambda\ and\ \theta$ can be obtained respectively as

$$\hat{\alpha}\pm z_{\frac{c}{2}}\sqrt{\hat{J}_{\alpha\alpha}},\qquad \hat{\lambda}\pm z_{\frac{c}{2}}\sqrt{\hat{J}_{\lambda\lambda}}\ and\ \hat{\theta}\pm z_{\frac{c}{2}}\sqrt{\hat{J}_{\theta\theta}},$$

Where $z_{\frac{c}{2}}$ is the upper c^{th} percentile of the standard normal distribution.

6 Applications

To illustrate the flexibility and tractability of the APTEBII model, we provide analysis to two real data sets. The first data set taken from Aarset [26] represents the failure times of 50 devices, while second data set taken from [27], represents the failure times of air-conditioned system of an airplane. We fit the APTEBII distribution and other five competing models namely; Alpha power transformed extended exponential (APTEE), alpha power transformed Weibull (APTW) [14], APTL, APTE, exponential (E) distributions and its sub-model. The two data sets are given in Table 4.

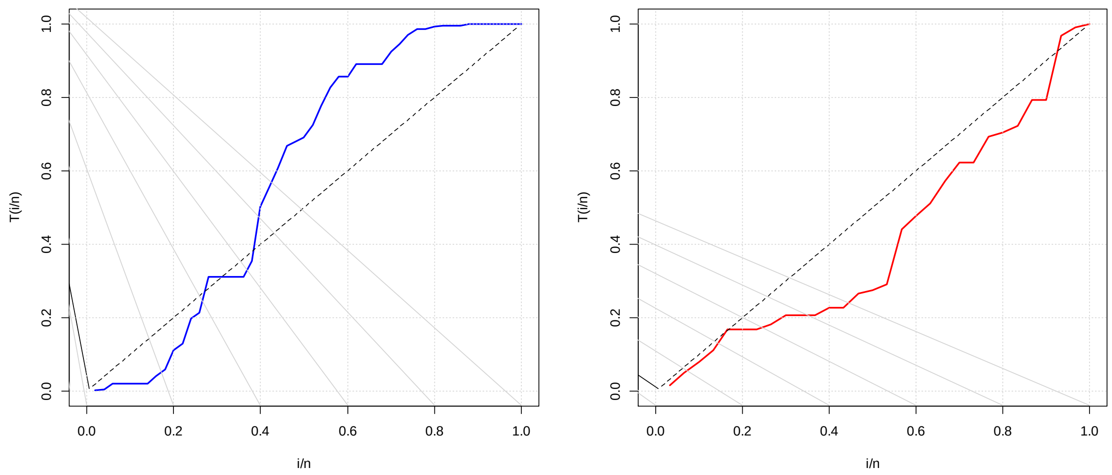
Table 4. The failure times of air-conditioned system of an airplane and Aarset

Data 1	0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82,83, 84, 84, 84, 85, 85, 85, 85,85, 86, 86
Data 2	23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120,11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95

Descriptive statistics for the two data sets considered are presented in Table 5, which includes mean, median, variance, skewness, among others. The graphs of total test time (TTT curves) to these data are presented in Fig. 5.

Table 5. Descriptive statistics for the data sets

Statistic	Data set I	Data set II
N	50	44
Mean	45.69	223.50
Median	48.50	128.50
Variance	1078.15	93286.41
Skewness	-0.1421	3.5044
Kurtosis	-1.6267	15.8667
Minimum	0.1	12.20
Maximum	86	1776
Lower quartile	13.50	67.21
Upper quartile	81.25	219.0



(a) TTT plot curve to data set I **(b) TTT plot curve to data set II**

Fig. 5. The graphs of total test time (TTT curves)

For all the fitted models, we compute the MLEs of the model parameters (with their corresponding standard errors in parentheses) and also the values of the Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC) , Bayesian Information Criterion (BIC) and Kolmogorov-Smirnoff (KS) statistic used as methods of comparing fits of distributions to data. In general, it is considered that the smaller the values of this statistic the better the model fit to the data.

Table 6 present the results related to the first data set which lists the MLEs of the model parameters (with the corresponding standard errors and the confidence intervals in parentheses) and the values of the values of the AIC, HQIC and the KS test statistics. These figures in this Tables reveals that the APTEBII model has the lowest AIC, HQIC, BIC and in terms of the KS statistics the APTEE distribution has the lowest.

Table 7 present the results related to the second data set which lists the MLEs of the model parameters (with the corresponding standard errors and the confidence intervals in parentheses) and the values of the values of the AIC, HQIC and the KS test statistics. These figures in this Table reveal that the APTEBII model has the lowest AIC, HQIC, BIC and in terms of the KS statistics the APTEE distribution has the lowest.

Table 6. MLEs, standard errors (in parentheses), AIC, HQIC and BIC values for the data set I

Model	MLE's	$-l$	AIC	BIC	$HQIC$	KS
APTEBII	$\hat{\alpha}$ =0.164(0.031) $\hat{\beta}$ =2.803(0.582) $\hat{\lambda}$ =3.596(0.722)	169.81	345.62	351.356	347.804	0.8115
APTEE	$\hat{\alpha}$ =2.11(1.643) $\hat{\beta}$ =0.021 $\times 10^{-4}$(0.024) $\hat{\lambda}$ =0.035(6.552 $\times 10^{-3}$)	281.447	568.893	567.990	571.078	0.17419
APTW	$\hat{\alpha}$ =8.911(9.1374) $\hat{\beta}$ =0.685(0.128) $\hat{\lambda}$ =0.121(0.078)	281.962	569.928	569.026	572.113	0.17423
APTE	$\hat{\alpha}$ =4.822 $\times 10^{-3}$ (3.528 $\times 10^{-6}$) $\hat{\lambda}$ =1.361 $\times 10^{-3}$ (6.19 $\times 10^{-3}$)	283.583	571.167	570.565	572.623	0.193111
APTL	$\hat{\alpha}$ =**4.359** $\times 10^{-6}$ (**6.762** $\times 10^{-3}$) $\hat{\lambda}$ =**8.754** $\times 10^{-3}$) (**6.886** $\times 10^{-3}$)	267.747	574.495	569.892	598.951	0.2186
E	$\hat{\lambda}$ = 4.204 $\times 10^{-4}$ (5.9448 $\times 110^{-5}$)	389.68	781.360	789.059	697.422	0.9645
BII	$\hat{\beta}$ =3.23(0.630) $\hat{\lambda}$ =3.081(0.629)		1446.292	1450.122	1447.755	0.8107

Table 7. MLEs, standard errors (in parentheses), AIC, HQIC and BIC values for the data set 2

Model	MLE's	$-2l$	AIC	BIC	$HQIC$	KS
APTEBII	$\hat{\alpha}$ =0.187(0.0417) $\hat{\beta}$ =45.210(8.270) $\hat{\lambda}$ =24.210(0.4.404)	-91.972	-85.973	-81.769	-84.628	0.9667
APTEE	$\hat{\alpha}$ =**0.161**(**0.282**) $\hat{\beta}$ =**2.01** $\times 10^{-4}$ (**0.024**) $\hat{\lambda}$ =**0.011**(**0.022**)	176.631	359.262	357.694	360.607	0.14683
APTW	$\hat{\alpha}$ =**6.257** $\times 10^{-10}$ (**9.854** $\times 10^{-8}$) $\hat{\beta}$ =**0.509**(**0.095**) $\hat{\lambda}$ =**7.168** $\times$ **10-3**(**0.057**)	182.718	371.436	369.867	372.78	0.26544
APTE	$\hat{\alpha}$ =**8.688** $\times 10^{-10}$ (**5.698** $\times 10^{-8}$) $\hat{\lambda}$ =**8.536** $\times 10^{-4}$ (**2.844** $\times 10^{-3}$)	`177.388	358.775	357.729	359.220	0.19989
APTL	$\hat{\alpha}$ =**0.1**(**0.104**) $\hat{\lambda}$ =**0.024**(**5.127** $\times 10^{-3}$)	202.837	407.674	407.159	408.123	0.663
E	$\hat{\lambda}$ =**4.339**$\times 10^{-4}$ (**7.922** $\times 10^{-5}$)	233.056	468.111	467.589	468.560	0.8929
BII	$\hat{\beta}$ =45.514(8.310) $\hat{\lambda}$ =30.684(5.602)	308.422	620.8439	623.646	621.7404	0.9667

7 Conclusion

In this paper we have proposed a new three-parameter family of distribution, called the APTEBII distribution. The proposed APTEBII model has two shape parameters and one scale parameter. The APTEBII density function can take various forms depending on its shape parameters. . We fit the APTEBII distribution and other five competing models namely; Alpha power transformed extended exponential distribution, alpha power transformed Weibull distribution, Alpha power transformed Lomax distribution, Alpha power exponential distribution, exponential (E) distributions and its sub-model. In modelling the two life data set presented in this work the Alpha power transformed extended Bur II distribution can also be used because of it flexibility.

Disclaimer

The products used for this research are commonly and predominantly use products in our area of research and country. There is absolutely no conflict of interest between the authors and producers of the products because we do not intend to use these products as an avenue for any litigation but for the advancement of knowledge. Also, the research was not funded by the producing company rather it was funded by personal efforts of the authors.

Competing Interests

Authors have declared that no competing interests exist.

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